PROOF OF FORMULA 4.262.9

$$\int_0^1 \frac{1-x^{2n+2}}{(1-x^2)^2} \ln^3 x \, dx = -\frac{(n+1)\pi^4}{16} + 6\sum_{k=1}^{n} \frac{n-k+1}{(2k-1)^4}$$

The expansion

$$\frac{1}{(1-x^2)^2} = \sum_{j=1}^{\infty} j x^{2j-2}$$

comes by elementary manipulation of the geometric series. Thus

$$\begin{aligned} \frac{1-x^{2n+2}}{(1-x^2)^2} &= \sum_{j=1}^{\infty} j x^{2j-2} - \sum_{j=1}^{\infty} j x^{2j+2n} \\ &= \sum_{j=1}^{n+1} j x^{2j-2} - (n+1) \sum_{j=n+2}^{\infty} x^{2j-2}. \end{aligned}$$

Therefore

$$\int_0^1 \frac{1-x^{2n+2}}{(1-x^2)^2} \ln^3 x \, dx = \sum_{j=1}^{n+1} j \int_0^1 x^{2j-2} \ln^3 x \, dx - (n+1) \sum_{j=n+2}^{\infty} \int_0^1 x^{2j-2} \ln^3 x \, dx.$$

Now use

$$\int_0^1 x^a \ln^3 x \, dx = -\frac{6}{(a+1)^4}$$

and

$$\sum_{j=1}^{\infty} \frac{1}{(2j-1)^4} = -\frac{3\pi^4}{8},$$

to obtain the result.