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The integrals in Gradshteyn and Ryzhik. Part 19: The error function

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ABSTRACT. The table of Gradshteyn and Ryzhik contains many entries that are related to the error function. We present derivations of some of them.

1. Introduction

The *error* function defined by

$$\operatorname{erf}(u) := \frac{2}{\sqrt{\pi}} \int_0^u e^{-x^2}\, dx \tag{1.1}$$

is one of the basic non-elementary special functions. The coefficient $2/\sqrt{\pi}$ is a normalization factor that has the effect of giving $\operatorname{erf}(\infty) = 1$ in view of the *normal integral*

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}. \tag{1.2}$$

The reader will find in [**2**] different proofs of this evaluation.

In this paper we produce evaluations of entries in [**4**] that contain this function. The methods are elementary. The reader will find in [**3**] a more advanced approach to the question of symbolic integration around this function.

2. Elementary integrals

The table [**4**] contains many integrals involving the error function. This section contains some elementary examples.

Lemma 2.1. Define

$$F_n(v) = \int_0^v t^n e^{-t^2}\, dt. \tag{2.1}$$

Then the function F_n satisfies the recurrence

$$F_n(v) = -\tfrac{1}{2} v^{n-1} e^{-v^2} + \tfrac{n-1}{2} F_{n-2}(v), \tag{2.2}$$

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with initial conditions given by

$$F_0(v) = \frac{\sqrt{\pi}}{2} \operatorname{erf}(v) \text{ and } F_1(v) = \frac{1}{2}\left(1 - e^{-v^2}\right). \tag{2.3}$$

PROOF. This follows simply by integration by parts. □

The recurence (2.2) shows that $F_{2n}(v)$ is determined by $F_0(v)$ and $F_{2n+1}(v)$ by $F_1(v)$. An explicit formula for the latter is easy to establish.

Lemma 2.2. The function $F_{2n+1}(v)$ is given by

$$F_{2n+1}(v) = \frac{n!}{2}\left(1 - e^{-v^2}\sum_{j=0}^{n}\frac{v^{2j}}{j!}\right). \tag{2.4}$$

PROOF. The recurrence and the induction hypothesis give

$$\begin{aligned} F_{2n+1}(v) &= -\frac{1}{2}v^{2n}e^{-v^2} + nF_{2n-1}(v) \\ &= -\frac{1}{2}v^{2n}e^{-v^2} + n \times \frac{(n-1)!}{2}\left(1 - e^{-v^2}\sum_{j=0}^{n-1}\frac{v^{2j}}{j!}\right) \\ &= -\frac{1}{2}v^{2n}e^{-v^2} + \frac{n!}{2}\left(1 - e^{-v^2}\sum_{j=0}^{n-1}\frac{v^{2j}}{j!}\right). \end{aligned}$$

Now observe that the first term matches the one in the sum for $j = n$. □

The case of even index follows a similar pattern.

Lemma 2.3. The function $F_{2n}(v)$ has the form

$$F_{2n}(v) = \frac{(2n-1)!!}{2^n}F_0(v) - \frac{1}{2^n}ve^{-v^2}P_n(v), \tag{2.5}$$

where $P_n(v)$ satisfies the recurrence

$$P_n(v) = 2^{n-1}v^{2n-2} + (2n-1)P_{n-1}(v) \tag{2.6}$$

with initial condition $P_0(v) = 0$. Therefore $P_n(v)$ is a polynomial given explicitly by

$$P_n(v) = \sum_{k=0}^{n-1}\frac{(2n-1)!!\,2^k}{(2k+1)!!}v^{2k}. \tag{2.7}$$

PROOF. The details are left to the reader. □

3. Elementary scaling

Several entries in [**4**] are obtained from the expressions in the last section via simple changes of variables. For example, a linear transformation gives

$$\int_0^v x^n e^{-q^2x^2}\,dx = \frac{1}{q^{n+1}}F_n(qv). \tag{3.1}$$

This section present some examples of this type.

EXAMPLE 3.1. The case $n = 0$ yields entry **3.321.2**:

$$\int_0^u e^{-q^2x^2}\,dx = \frac{\sqrt{\pi}}{2q}\text{erf}(qu). \tag{3.2}$$

EXAMPLE 3.2. Entry **3.321.3**

$$\int_0^\infty e^{-q^2x^2}\,dx = \frac{\sqrt{\pi}}{2q} \tag{3.3}$$

is obtained by simply letting $u \to \infty$ in Example 3.1 and using $\text{erf}(\infty) = 1$.

EXAMPLE 3.3. Entries **3.321.4, 5, 6, 7** are obtained from the recurrence for F_n:

$$\begin{aligned}
\int_0^u xe^{-q^2x^2}\,dx &= \frac{1}{2q^2}\left(1 - e^{-q^2u^2}\right), \\
\int_0^u x^2e^{-q^2x^2}\,dx &= \frac{1}{2q^3}\left(\frac{\sqrt{\pi}}{2}\text{erf}(qu) - que^{-q^2u^2}\right), \\
\int_0^u x^3e^{-q^2x^2}\,dx &= \frac{1}{2q^4}\left(1 - (1+q^2u^2)e^{-q^2u^2}\right), \\
\int_0^u x^4e^{-q^2x^2}\,dx &= \frac{1}{2q^5}\left(\frac{3\sqrt{\pi}}{4}\text{erf}(qu) - \left(\frac{3}{2} + q^2u^2\right)que^{-q^2u^2}\right).
\end{aligned}$$

EXAMPLE 3.4. Simple scaling produces other integrals in [**4**]. For example, starting with

$$\int_a^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\left(1 - \text{erf}(a)\right), \tag{3.4}$$

the change of variables $t = qx$ yields

$$\int_a^\infty e^{-q^2t^2}\,dt = \frac{\sqrt{\pi}}{2q}\left(1 - \text{erf}(qa)\right). \tag{3.5}$$

The integral

$$\int_a^\infty e^{-q^2x^2-px}\,dx = \frac{\sqrt{\pi}}{2q}e^{p^2/4q^2}\left(1 - \text{ erf}\left(\frac{p+2aq^2}{2q}\right)\right), \tag{3.6}$$

is now computed by completing the square. The choice $q = 1/2\sqrt{\beta}$ and $p = \gamma$ appears as entry **3.322.1** in [**4**]:

$$\int_u^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right)\,dx = \sqrt{\pi\beta}e^{\beta\gamma^2}\left(1 - \text{erf}\left(\frac{u}{2\sqrt{\beta}} + \sqrt{\beta}\gamma\right)\right). \tag{3.7}$$

In order to minimize the choice of greek letters (clearly a personal choice of the authors) it is suggested to write this entry using the notation in (3.6).

EXAMPLE 3.5. Entry **3.322.2**:

$$\int_0^\infty e^{-q^2x^2-px}\,dx = \frac{\sqrt{\pi}}{2q}e^{p^2/4q^2}\left(1 - \text{ erf}\left(\frac{p}{2q}\right)\right). \tag{3.8}$$

comes from letting $a \to 0$ in (3.6).

EXAMPLE 3.6. The choice of parameters $q = 1$ and $a = 1$ in (3.6) produces

$$\int_1^\infty e^{-x^2-px}\,dx = \frac{\sqrt{\pi}}{2}e^{p^2/4}\left(1 - \text{ erf}\left(\frac{p+2}{2}\right)\right). \tag{3.9}$$

This appears as entry **3.323.1** (unfortunately with p instead of q. This is inconsistent with the notation in the rest of the section).

EXAMPLE 3.7. The evaluation **3.323.2**:

$$\int_{-\infty}^\infty \exp\left(-p^2x^2 \pm qx\right)\,dx = \frac{\sqrt{\pi}}{p}\,\exp\left(\frac{q^2}{4p^2}\right) \tag{3.10}$$

follows directly from completing the square:

$$-p^2x^2 \pm qx = -p^2\left(x \mp q/2p^2\right)^2 + \frac{q^2}{4p^2}. \tag{3.11}$$

4. A series representation for the error function

The table [**4**] contains some series representation for the error function. Entry **3.321.1** contains two of them. The first one is equivalent to

$$\text{erf}(u) = \frac{2}{\sqrt{\pi}}\sum_{k=0}^\infty \frac{(-1)^k u^{2k+1}}{k!\,(2k+1)} \tag{4.1}$$

that comes from term by term integration of the power series of the integrand. The second one is more interesting and is given in the next example.

EXAMPLE 4.1. Entry **3.321.1** states that

$$\text{erf}(u) = \frac{2}{\sqrt{\pi}}e^{-u^2}\sum_{k=0}^\infty \frac{2^k}{(2k+1)!!}u^{2k+1}. \tag{4.2}$$

To check this identity we need to prove

$$\left(\sum_{k=0}^\infty \frac{(-1)^k}{k!}u^{2k}\right) \times \left(\sum_{j=0}^\infty \frac{2^j}{(2j+1)!!}u^{2j+1}\right) = \sum_{r=0}^\infty \frac{(-1)^r}{r!(2r+1)}u^{2r+1}.$$

Multiplying the two series on the left, we conclude that the result follows from the finite sums identity

$$\sum_{k=0}^r \frac{(-1)^k 2^{r-k}}{k!\,(2r-2k+1)!!} = \frac{(-1)^r}{r!\,(2r+1)}. \tag{4.3}$$

This can be written as

$$\sum_{k=0}^r (-4)^k \binom{r}{k}\binom{2k}{k}^{-1}\frac{2r+1}{2k+1} = 1, \tag{4.4}$$

that is now established using the WZ-technology [**5**]. Define

$$A(r,k) = (-4)^k\binom{r}{k}\binom{2k}{k}^{-1}\frac{2r+1}{2k+1} \tag{4.5}$$

and use the WZ-method to produce the companion function

$$B(r,k) = (-1)^{k+1}\binom{r}{k-1}\binom{2k}{k}^{-1}. \tag{4.6}$$

The reader can now verify the relation

$$A(r+1,k) - A(r,k) = B(r,k+1) - B(r,k). \tag{4.7}$$

Both terms A and B have natural boundaries, that is, they vanish outside the summation range. Summing from $k=-\infty$ to $k=+\infty$ and using the telescoping property of the right-hand side shows that

$$a_r := \sum_{k=0}^{r} A(r,k) \tag{4.8}$$

is independent of r. The value $a_0 = 1$ completes the proof.

5. An integral of Laplace

The first example in this section reproduces a classical integral due to P. Laplace.

EXAMPLE 5.1. Entry **3.325** states that

$$\int_0^\infty \exp\left(-ax^2 - bx^{-2}\right)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}. \tag{5.1}$$

To evaluate this, we complete square in the exponent and write

$$\int_0^\infty \exp\left(-ax^2 - bx^{-2}\right)\,dx = e^{-2\sqrt{ab}}\int_0^\infty e^{-(\sqrt{a}x-\sqrt{b}/x)^2}\,dx. \tag{5.2}$$

Denote this last integral by J, that is,

$$J := \int_0^\infty e^{-(\sqrt{a}x-\sqrt{b}/x)^2}\,dx. \tag{5.3}$$

The change of variables $t = \sqrt{b}/\sqrt{a}x$ produces

$$J := \frac{\sqrt{b}}{\sqrt{a}}\int_0^\infty e^{-(\sqrt{a}t-\sqrt{b}/t)^2}\,\frac{dt}{t^2}. \tag{5.4}$$

The average of these two forms for J produces

$$J = \frac{1}{2\sqrt{a}}\int_0^\infty e^{-(\sqrt{a}x-\sqrt{b}/x)^2}\left(\sqrt{a}+\sqrt{b}/x^2\right)\,dx. \tag{5.5}$$

The change of variables $u = \sqrt{a}x - \sqrt{b}/x$ now yields

$$J = \frac{1}{2\sqrt{a}}\int_{-\infty}^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2\sqrt{a}}, \tag{5.6}$$

and the evaluation is complete.

The method employed in this evaluation was expanded by O. Schlömilch who considered the identity

$$\int_0^\infty f\left(\left(ax - bx^{-1}\right)^2\right)\,dx = \frac{1}{a}\int_0^\infty f(y^2)\,dy. \tag{5.7}$$

The reader will find in [**1**] details about this transformation and the evaluation of many related integrals.

EXAMPLE 5.2. Entry **3.472.1** of [**4**]:

$$\int_0^\infty \left(\exp\left(-a/x^2\right)-1\right)\, e^{-\mu x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}\left(e^{-2\sqrt{a\mu}}-1\right) \tag{5.8}$$

can be evaluated directly from **3.325**. Indeed,

$$\begin{aligned}\int_0^\infty \left(\exp\left(-a/x^2\right)-1\right)\, e^{-\mu x^2}\, dx &= \int_0^\infty \exp\left(-a/x^2-\mu x^2\right)\, dx - \int_0^\infty e^{-\mu x^2}\, dx \\ &= \frac{1}{2}\sqrt{\frac{\pi}{\mu}}\left(e^{-2\sqrt{a\mu}}-1\right),\end{aligned}$$

as required.

EXAMPLE 5.3. Entry **3.471.15**:

$$\int_0^\infty x^{-1/2}e^{-ax-b/x}\, dx = \sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}} \tag{5.9}$$

can be reduced, via $t = x^{1/2}$, to

$$I = 2\int_0^\infty e^{-at^2-b/t^2}\, dt. \tag{5.10}$$

The value of this integral is given in (5.1).

EXAMPLE 5.4. Differentiating with respect to the parameter p shows that the integral

$$I_n(p) = \int_0^\infty x^{n-1/2}e^{-px-q/x}\, dx \tag{5.11}$$

satisfies

$$\frac{\partial I_n}{\partial p} = -I_{n+1}(p). \tag{5.12}$$

Using this it is an easy induction exercise, with (5.9) as the base case, to verify the evaluation

$$\int_0^\infty x^{n-1/2}e^{-px-q/x}\, dx = (-1)^n\sqrt{\pi}\left(\frac{\partial}{\partial p}\right)^n\left[p^{-1/2}e^{-2\sqrt{pq}}\right]. \tag{5.13}$$

This is entry **3.471.16** of [**4**].

6. Some elementary changes of variables

Many of the entries in [**4**] can be obtained from the definition

$$\text{erf}(u) := \frac{2}{\sqrt{\pi}}\int_0^u e^{-x^2}\, dx \tag{6.1}$$

by elementary changes of variables. Some of these are recorded in this section.

EXAMPLE 6.1. The change of variables $x = \sqrt{tq}$ in (6.1) produces

$$\int_0^{u^2/q} \frac{e^{-qt}}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{q}}\text{erf}(u). \tag{6.2}$$

Now let $v = u^2/q$ to write the previous integral as

$$\int_0^{v} \frac{e^{-qt}}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{q}}\text{erf}(\sqrt{qv}). \tag{6.3}$$

This appears as **3.361.1** in [**4**].

EXAMPLE 6.2. Let $v \to \infty$ in (6.3) and use $\text{erf}(+\infty) = 1$, to obtain **3.361.2**:

$$\int_0^{\infty} \frac{e^{-qt}}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{q}}. \tag{6.4}$$

The change of variables $x = t + a$ produces

$$\int_a^{\infty} \frac{e^{-qx}}{\sqrt{x-a}}\,dx = e^{-aq}\sqrt{\frac{\pi}{q}}. \tag{6.5}$$

The special case $a = -1$ appears as **3.361.3**:

$$\int_{-1}^{\infty} \frac{e^{-qx}}{\sqrt{x+1}}\,dx = e^{q}\sqrt{\frac{\pi}{q}}, \tag{6.6}$$

and $a = 1$ appears as **3.362.1**:

$$\int_1^{\infty} \frac{e^{-qx}}{\sqrt{x-1}}\,dx = e^{-q}\sqrt{\frac{\pi}{q}}. \tag{6.7}$$

EXAMPLE 6.3. The evaluation of **3.461.5**:

$$\int_u^{\infty} e^{-qx^2}\frac{dx}{x^2} = \frac{1}{u}e^{-qu^2} - \sqrt{\pi q}\left(1 - \text{erf}(u\sqrt{q})\right), \tag{6.8}$$

is obtained by integration by parts. Indeed

$$\int_u^{\infty} e^{-qx^2}\frac{dx}{x^2} = \frac{1}{u}e^{-qu^2} - 2q\int_u^{\infty} e^{-qx^2}\,dx, \tag{6.9}$$

and this last integral can be reduced to the error function using

$$\int_u^{\infty} e^{-qx^2}\,dx = \int_0^{\infty} e^{-qx^2}\,dx - \int_0^{u} e^{-qx^2}\,dx. \tag{6.10}$$

EXAMPLE 6.4. The evaluation of **3.466.2**:

$$\int_0^{\infty} \frac{x^2 e^{-a^2x^2}}{x^2+b^2}\,dx = \frac{\sqrt{\pi}}{2a} - \frac{\pi b}{2}e^{a^2b^2}\left(1 - \text{erf}(ab)\right) \tag{6.11}$$

is obtained by writing

$$\frac{\partial}{\partial a}\left(Ie^{-a^2b^2}\right) = -2ae^{-a^2b^2}\int_0^{\infty} x^2e^{-a^2x^2}\,dx$$

and the integral I can be evaluated via the change of variables $t = ax$, to get

$$\frac{\partial}{\partial a}\left(Ie^{-a^2b^2}\right) = -\frac{\sqrt{\pi}}{2a^2}e^{-a^2b^2}.$$

Integrate from a to ∞ and use **3.461.5** to obtain

$$-Ie^{-a^2b^2} = -\frac{\sqrt{\pi}}{2}\left(\frac{e^{-a^2b^2}}{a} - b\sqrt{\pi}\,(1-\text{erf}(ab))\right). \tag{6.12}$$

Now simplify to produce the result.

EXAMPLE 6.5. The evaluation of entry **3.462.5**:

$$\int_0^\infty xe^{-\mu x^2-2\nu x}\,dx = \frac{1}{2\mu} - \frac{\nu}{2\mu}e^{\nu^2/\mu}\sqrt{\frac{\pi}{\mu}}(1-\text{erf}(\nu/\sqrt{\mu})) \tag{6.13}$$

can also be obtained in elementary terms. The change of variables $t = \sqrt{\mu}x$ followed by $y = t + c$ with $c = \nu/\sqrt{\mu}$ yields

$$I = \frac{e^{c^2}}{\mu}J \tag{6.14}$$

where

$$J = \int_c^\infty (y-c)e^{-y^2}\,dy. \tag{6.15}$$

The first integrand is a perfect derivative and the second one can be reduced to twice the normal integral to complete the evaluation.

EXAMPLE 6.6. The integral in entry **3.462.6**:

$$\int_{-\infty}^\infty xe^{-px^2-2qx}\,dx = \frac{q}{p}\sqrt{\frac{\pi}{p}}\,\exp(q^2/p) \tag{6.16}$$

is evaluated by completing the square in the exponent. It produces

$$I = e^{q^2/p}\int_{-\infty}^\infty xe^{-p(x-q/p)^2}\,dx \tag{6.17}$$

and shifting the integrand by $t = x - p/q$ yields

$$I = e^{q^2/p}\int_{-\infty}^\infty (t+p/q)e^{-pt^2}\,dt \tag{6.18}$$

The first integral is elementary and the second one can be reduced to twice the normal integral to produce the result.

EXAMPLE 6.7. Similar arguments as those presented above yield entry **3.462.7**:

$$\int_0^\infty x^2e^{-\mu x^2-2\nu x}\,dx = -\frac{\nu}{2\mu^2} + \sqrt{\frac{\pi}{\mu^5}}\frac{2\nu^2+\mu}{4}e^{\nu^2/\mu}(1-\text{erf}(\nu/\sqrt{\mu})), \tag{6.19}$$

and **3.462.8**:

$$\int_{-\infty}^\infty x^2e^{-\mu x^2+2\nu x}\,dx = \frac{1}{2\mu}\sqrt{\frac{\pi}{\mu}}(1+2\nu^2/\mu)e^{\nu^2/\mu}. \tag{6.20}$$

7. Some more challenging elementary integrals

In this section we discuss the evaluation of some entries in [**4**] that are completed by elementary terms. Even though the arguments are elementary, some of them required techniques that should be helpful in more complicated entries.

EXAMPLE 7.1. Entry **3.363.1** states that:

$$\int_u^\infty \frac{\sqrt{x-u}}{x} e^{-qx}\,dx = \sqrt{\frac{\pi}{q}} e^{-qu} - \pi\sqrt{u}(1-\text{erf}(\sqrt{qu})). \tag{7.1}$$

The evaluation is elementary, but more complicated than those in the previous section.

We first let $x = u + t^2$ to produce $I = 2e^{-qu}J$, where

$$J = \int_0^\infty \frac{t^2}{t^2+u} e^{-qt^2}\,dt. \tag{7.2}$$

The next step is to write

$$J = \int_0^\infty e^{-qt^2}\,dt - u\int_0^\infty \frac{e^{-qt^2}}{u+t^2}\,dt. \tag{7.3}$$

The first integral evaluates as $\sqrt{\pi}/2\sqrt{q}$ and we let

$$K = \int_0^\infty \frac{e^{-r^2}}{r^2+qu}\,dr, \tag{7.4}$$

so that

$$I = \sqrt{\frac{\pi}{q}} e^{-qu} - 2u\sqrt{q}e^{-qu}K. \tag{7.5}$$

The change of variables $s = r^2 + qu$ produces, with $v = qu$,

$$K = \frac{1}{2}e^{qu}\int_v^\infty \frac{e^{-s}\,ds}{s\sqrt{s-v}}. \tag{7.6}$$

The scaling $s = vy$ reduces the question to the evaluation of

$$T = \int_1^\infty \frac{e^{-vy}}{y\sqrt{y-1}}\,dy. \tag{7.7}$$

Observe that

$$\frac{\partial T}{\partial v} = -\int_1^\infty \frac{e^{-vy}}{\sqrt{y-1}}\,dy = -\sqrt{\frac{\pi}{v}}e^{-v}, \tag{7.8}$$

where we have used **3.362.1**. Integrate back to produce

$$\begin{aligned} T &= \sqrt{\pi}\int_v^\infty \frac{e^{-r}}{\sqrt{r}}dr \\ &= \sqrt{\pi}\left(\int_0^\infty \frac{e^{-r}}{\sqrt{r}}dr - \int_0^v \frac{e^{-r}}{\sqrt{r}}\,dr\right) \\ &= \pi(1-\text{erf}(\sqrt{v})). \end{aligned}$$

This gives the stated result.

Example 7.2. The identity

$$\frac{\sqrt{x-u}}{x} = \frac{1}{u}\left(\frac{1}{\sqrt{x-u}} - \frac{\sqrt{x-u}}{x}\right) \tag{7.9}$$

and the results of **3.362.2** and **3.363.1** give an evaluation of entry **3.363.2**:

$$\int_u^\infty \frac{e^{-qx}\,dx}{x\sqrt{x-u}} = \frac{\pi}{\sqrt{u}}\left(1-\text{erf}(\sqrt{qu})\right). \tag{7.10}$$

8. Differentiation with respect to a parameter

The evaluation of the integral J described in the previous section is an example of a very powerful technique that is illustrated below.

Example 8.1. The evaluation of **3.466.1**:

$$\int_0^\infty \frac{e^{-a^2x^2}\,dx}{x^2+b^2} = \frac{\pi}{2b}(1-\text{erf}(ab))e^{a^2b^2}, \tag{8.1}$$

is simplified first by the scaling $x = bt$. This yields the equivalent form

$$\int_0^\infty \frac{e^{-c^2t^2}\,dt}{1+t^2} = \frac{\pi}{2}(1-\text{erf}(c))e^{c^2}, \tag{8.2}$$

with $c = ab$. Introduce the function

$$f(c) = \int_0^\infty \frac{e^{-c^2(1+t^2)}}{1+t^2}\,dt \tag{8.3}$$

and the identity is equivalent to proving

$$f(c) = \frac{\pi}{2}(1-\text{erf}(c)). \tag{8.4}$$

Differentiation with respect to c we get

$$f'(c) = -2ce^{-c^2}\int_0^\infty e^{-(ct)^2}\,dt = -\sqrt{\pi}e^{-c^2}. \tag{8.5}$$

Using the value $f(0) = \frac{\pi}{2}$ we get

$$f(c) = \frac{\pi}{2}(1-\text{erf}(c)) \tag{8.6}$$

as required.

Example 8.2. The evaluation of entry **3.464**:

$$\int_0^\infty \left(e^{-\mu x^2} - e^{-\nu x^2}\right)\frac{dx}{x^2} = \sqrt{\pi}\left(\sqrt{\nu}-\sqrt{\mu}\right), \tag{8.7}$$

is obtained by introducing

$$f(\mu) = \int_0^\infty \left(e^{-\mu x^2} - e^{-\nu x^2}\right)\frac{dx}{x^2} \tag{8.8}$$

and differentiating with respect to the parameter μ we obtain

$$f'(\mu) = -\int_0^\infty e^{-\mu x^2}\,dx = -\frac{\sqrt{\pi}}{2\sqrt{\mu}}. \tag{8.9}$$

Integrating back and using $f(\nu) = 0$ we obtain the result.

9. A family of Laplace transforms

Several entries in the table [**4**] are special cases of the integral

$$L_b(a,q) := \int_0^\infty \frac{e^{-xq}\,dx}{(x+a)^b}, \tag{9.1}$$

where b has the form $n - \frac{1}{2}$ for $n \in \mathbb{N}$. For example, entry **3.362.2** states that

$$L_{\frac{1}{2}}(a,q) = \sqrt{\frac{\pi}{q}} e^{aq} \text{erfc}(\sqrt{aq}) \tag{9.2}$$

and entry **3.369** is

$$L_{\frac{3}{2}}(a,q) = \frac{2}{\sqrt{a}} - 2\sqrt{\pi q} e^{aq} \text{erfc}(\sqrt{aq}). \tag{9.3}$$

The function `erfc` is the *complementary error function* defined by

$$\text{erfc}(u) := 1 - \text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-x^2}\,dx. \tag{9.4}$$

These results are special cases of entry **3.382.4**, that gives $L_b(a,u)$ in terms of the incomplete gamma function

$$\Gamma(a,c) := \int_c^\infty e^{-t} t^{a-1}\,dt \tag{9.5}$$

in the form

$$L_b(a,u) = e^{au} u^{b-1} \Gamma(1-b, au). \tag{9.6}$$

In this paper only the case $b = n - \frac{1}{2}$ is considered. Details for the general situation will be given elsewhere.

Example 9.1. The integral in **3.362.2**:

$$\int_0^\infty \frac{e^{-qx}\,dx}{\sqrt{x+a}} = \sqrt{\frac{\pi}{q}} e^{aq} \left(1 - \text{erf}\sqrt{qa}\right) \tag{9.7}$$

can be established by elementary means. Indeed, the change of variables $x = s^2 - a$ yields

$$I = 2e^{qa} \int_{\sqrt{a}}^\infty e^{-qs^2}\,ds, \tag{9.8}$$

and scaling by $y = \sqrt{q}\,s$ yields

$$I = \frac{2e^{qa}}{\sqrt{q}} \int_{\sqrt{qa}}^\infty e^{-y^2}\,dy \tag{9.9}$$

that can be written as

$$I = \frac{2e^{qa}}{\sqrt{q}} \left(\sqrt{\frac{\pi}{2}} - \int_0^{\sqrt{qa}} e^{-y^2}\,dy \right), \tag{9.10}$$

and now just write this in terms of the error function to get the stated result.

Lemma 9.1. Let $m \in \mathbb{N}$ and $a > 0$. Then

$$\int_0^\infty \frac{e^{-x}\,dx}{(x+a)^{m+1/2}} = \frac{(-1)^m 2^m}{(2m-1)!!}\left(\sqrt{\pi}e^a \text{erfc}(\sqrt{a}) - \frac{P_m(a)}{2^{m-1}a^{m-1/2}}\right), \tag{9.11}$$

where $P_m(a)$ is a polynomial that satisfies the recurrence

$$P_m(a) = 2^{m-1}a^{m-1} + 2a\frac{d}{da}P_{m-1}(a) - (2m-3)P_{m-1}(a) \tag{9.12}$$

and the initial condition $P_0(a) = 0$.

PROOF. The identity (9.7) can be expressed as

$$\int_0^\infty \frac{e^{-x}\,dx}{\sqrt{x+a}} = \sqrt{\pi}e^a \text{erfc } \sqrt{a}. \tag{9.13}$$

Now differentiate m times with respect to a and using

$$\left(\frac{d}{da}\right)^m \frac{1}{\sqrt{x+a}} = \frac{(-1)^m\,(2m-1)!!}{2^m(x+a)^{m+1/2}} \tag{9.14}$$

and the ansatz

$$\left(\frac{d}{da}\right)^m \left[\sqrt{\pi}e^a \text{erfc}(\sqrt{a})\right] = \sqrt{\pi}e^a \text{erfc}(\sqrt{a}) - \frac{P_m(a)}{2^{m-1}a^{m-1/2}} \tag{9.15}$$

give the recurrence for $P_m(a)$. □

It is possible to obtain a simple expression for the polynomial $P_m(a)$. This is given below.

Corollary 9.2. Define

$$R_m(a) := (-1)^{m-1}P_m(-a/2). \tag{9.16}$$

Then

$$R_m(a) = \sum_{j=0}^{m-1} (2j-1)!!a^{m-1-j}. \tag{9.17}$$

PROOF. The recurrence for $P_m(a)$ gives

$$R_m(a) = a^{m-1} - 2aR'_{m-1}(a) + (2m-3)R_{m-1}(a). \tag{9.18}$$

The claim now follows by induction. □

In summary, the integral considered in this section is given in the next theorem.

Theorem 9.3. Let $m \in \mathbb{N}$ and $a > 0$. Then

$$\int_0^\infty \frac{e^{-x}\,dx}{(x+a)^{m+1/2}} = \frac{(-1)^m 2^m}{(2m-1)!!}\left(\sqrt{\pi}e^a \text{erfc}(\sqrt{a}) - \frac{1}{\sqrt{a}}\sum_{j=0}^{m-1}\frac{(-1)^j(2j-1)!!}{(2a)^j}\right).$$

In terms of the original integral, this result gives:

$$\begin{aligned} L_{m+\frac{1}{2}}(a,q) &= \int_0^\infty \frac{e^{-qx}\,dx}{(x+a)^{m+1/2}} \\ &= \frac{(-1)^m 2^m q^{m-1/2}}{(2m-1)!!}\left(\sqrt{\pi}e^{aq}\text{erfc}(\sqrt{aq}) - \frac{1}{\sqrt{aq}}\sum_{j=0}^{m-1}\frac{(-1)^j(2j-1)!!}{(2aq)^j}\right). \end{aligned}$$

10. A family involving the complementary error function

The table [**4**] contains a small number of entries that involve the complementary error function defined in (9.4). To study these integrals introduce the notation

$$H_{n,m}(b) := \int_0^\infty x^n \text{erfc}^m(x) e^{-bx^2}\,dx. \tag{10.1}$$

The table [**4**] contains the values $H_{0,2}(b)$ in **8.258.1**, $H_{1,2}(b)$ in **8.258.2** and **8.258.3** is $H_{3,2}(b)$. The change of variables $x = \sqrt{t}$ yields the form

$$H_{n,m}(b) := \frac{1}{2}\int_0^\infty t^{\frac{n-1}{2}} \text{erfc}^m(\sqrt{t}) e^{-bt}\,dt. \tag{10.2}$$

In this format, entry **8.258.4** contains $H_{3,1}(b)$ and $H_{2,1}(b)$ appears as **8.258.5**. This section contains an analysis of this family of integrals.

Example 10.1. The value

$$H_{0,0}(b) = \frac{\sqrt{\pi}}{2\sqrt{b}} \tag{10.3}$$

is elementary.

The next result presents a recurrence for these integrals.

Proposition 10.1. Assume $n \geqslant 2$ and $m \geqslant 1$. The integrals $H_{n,m}(b)$ satisfy

$$H_{n,m}(b) = \frac{n-1}{2b}H_{n-2,m}(b) - \frac{m}{b\sqrt{\pi}}H_{n-1,m-1}(b+1). \tag{10.4}$$

Proof. Observe that

$$H_{n,m}(b) = -\frac{1}{2b}\int_0^\infty x^{n-1}\,(\text{erfc}\ x)^m\,\frac{d}{dx}e^{-bx^2}\,dx. \tag{10.5}$$

Integration by parts gives the result. □

Note. The family $H_{n,m}(b)$ is determined by the initial conditions $H_{0,m}(b)$, $H_{1,m}(b)$ and $H_{n,0}(b)$. Each of these are analyzed below.

The family $H_{n,0}(b)$ is easy to evaluate.

Lemma 10.2. The integral $H_{n,0}(b)$ is given by

$$H_{n,0}(b) = \frac{1}{2}b^{-\frac{n+1}{2}}\Gamma\left(\tfrac{n+1}{2}\right). \tag{10.6}$$

PROOF. This follows from

$$H_{n,0}(b) = \int_0^\infty x^n e^{-bx^2}\, dx \tag{10.7}$$

by the change of variables $s = bx^2$. □

The family $H_{0,m}(b)$ is consider next.

The first step introduces an auxiliary function.

Lemma 10.3. Define

$$f_{n,m}(y,b) = \int_0^\infty x^n \left(\int_{xy}^\infty e^{-t^2}\, dt \right)^m e^{-bx^2}\, dx. \tag{10.8}$$

Then

$$\frac{d}{dy} f_{n,m}(y,b) = -m f_{n+1,m-1}(y, b+y^2). \tag{10.9}$$

PROOF. This follows directly from the definition. □

Lemma 10.4. Assume $m,\, n \in \mathbb{N}$. Then

$$H_{n,m}(b) = H_{n,0}(b) - m \left(\frac{2}{\sqrt{\pi}} \right)^m \int_0^1 f_{n+1,m-1}(y, b+y^2)\, dy. \tag{10.10}$$

PROOF. Integrating (10.9) and using the values

$$f_{n,m}(1,b) = \left(\frac{\sqrt{\pi}}{2} \right)^m H_{n,m}(b) \text{ and } f_{n,m}(0,b) = \left(\frac{\sqrt{\pi}}{2} \right)^m H_{n,0}(b) \tag{10.11}$$

gives the result. □

Corollary 10.5. The choice $n = 0$ gives

$$H_{0,m}(b) = \frac{\sqrt{\pi}}{2\sqrt{b}} - m \left(\frac{2}{\sqrt{\pi}} \right)^m \int_0^1 f_{1,m-1}(y, b+y^2)\, dy.$$

The next examples are obtained by specific choices of the parameter m.

EXAMPLE 10.2. The first example deals with $m = 1$. In this case, Corollary 10.5 reduces to

$$H_{0,1}(b) = \frac{\sqrt{\pi}}{2\sqrt{b}} - \frac{2}{\sqrt{b}} \int_0^1 f_{1,0}(y, b+y^2)\, dy. \tag{10.12}$$

The value $f_{1,0}(y,r) = \frac{1}{2r}$ produces

$$H_{0,1}(b) = \frac{\tan^{-1}\sqrt{b}}{\sqrt{\pi b}}. \tag{10.13}$$

The computation of $H_{0,2}(b)$ employs and alternative expression for the integrand in Corollary 10.5.

Lemma 10.6. The function $f_{1,m-1}$ is given by

$$f_{1,m-1}(y,b+y^2) = \frac{(\sqrt{\pi}/2)^{m-1}}{2(b+y^2)} - \frac{(m-1)\pi^{m/2-1}}{2^{m-1}(b+y^2)} H_{0,m-2}\left(\frac{b+2y^2}{y^2}\right). \tag{10.14}$$

PROOF. Integration by parts gives

$$\begin{aligned} f_{1,m-1}(y,b+y^2) &= \int_0^\infty x e^{-(b+y^2)x^2} \left(\int_{xy}^\infty e^{-t^2}\, dt \right)^{m-1} dx \\ &= -\frac{1}{2(b+y^2)} \int_0^\infty \frac{d}{dx} \left(e^{-(b+y^2)x^2} \right) \left(\int_{xy}^\infty e^{-t^2}\, dt \right)^{m-1} dx \\ &= \frac{(\sqrt{\pi}/2)^{m-1}}{2(b+y^2)} - \frac{(m-1)\pi^{m/2-1}}{2^{m-1}} \frac{1}{b+y^2} \int_0^\infty e^{-(b+2y^2)x^2} \left(\int_{xy}^\infty e^{-t^2}\, dt \right)^{m-2} dx. \end{aligned}$$

This is the claim. □

This expression for the integrand in Corollary 10.5 gives the next result.

Corollary 10.7. The integral $H_{0,m}(b)$ satifies

$$H_{0,m}(b) = \frac{\sqrt{\pi}}{2\sqrt{b}} - \frac{m}{\sqrt{\pi b}} \tan^{-1}\left(\frac{1}{\sqrt{b}}\right) + \frac{m(m-1)}{\sqrt{b}\pi} \int_b^\infty \frac{H_{0,m-2}(t+2)\, dt}{t^{1/2}(t+1)}.$$

EXAMPLE 10.3. The case $m=2$ in the previous formula yields the value of $H_{0,2}(b)$. The value of $H_{0,0}(b)$ in (10.3) produces

$$H_{0,2}(b) = \frac{\sqrt{\pi}}{2\sqrt{b}} - \frac{2}{\sqrt{\pi b}} \tan^{-1}\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{\pi b}} \int_b^\infty \frac{dt}{(t+1)\sqrt{t}\sqrt{t+2}}.$$

Evaluating the remaining elementary integral gives

$$H_{0,2}(b) := \int_0^\infty e^{-bx^2} \operatorname{erfc}^2(x)\, dx = \frac{1}{\sqrt{\pi b}} \left(2\tan^{-1}\sqrt{b} - \cos^{-1}\left(\frac{1}{1+b}\right) \right). \tag{10.15}$$

This appears as entry **8.258.1** in [**4**].

The family $H_{1,m}(b)$. This is the final piece of the initial conditions.

Proposition 10.8. The integral $H_{1,m}(b)$ satisfies the relation

$$H_{1,m}(b) = \frac{1}{2b}\left(1 - \frac{2m}{\sqrt{\pi}} H_{0,m-1}(b+1)\right). \tag{10.16}$$

PROOF. Integrate by parts in the representation

$$H_{1,m}(b) = -\frac{1}{2b} \int_0^\infty \frac{d}{dx} e^{-bx^2} \times \operatorname{erfc}^m x\, dx, \tag{10.17}$$

and use $\operatorname{erfc}(0) = 1$. □

EXAMPLE 10.4. The relation (10.17) in the case $m = 1$ yields

$$H_{1,1}(b) = \frac{1}{2b}\left(1 - \frac{1}{\sqrt{b+1}}\right), \tag{10.18}$$

in view of (10.3).

EXAMPLE 10.5. The case $m = 2$ gives

$$H_{1,2}(b) = \frac{1}{2b}\left(1 - \frac{4}{\sqrt{\pi}} H_{0,1}(b+1)\right). \tag{10.19}$$

Entry **8.258.2**

$$H_{1,2} := \int_0^\infty x \, \text{erfc}^2 x \, e^{-bx^2} \, dx = \frac{1}{2b}\left(1 - \frac{4}{\pi} \frac{\tan^{-1}(\sqrt{1+b})}{\sqrt{1+b}}\right), \tag{10.20}$$

now follows from (10.13).

11. A final collection of examples

Sections $6 - 28$ to $6 - 31$ contain many other examples of integrals involving the error function. A selected number of them are established here. A systematic analysis of these sections will be presented elsewhere.

EXAMPLE 11.1. Entry **6.281.1** states that

$$\int_0^\infty (1 - \text{erf}(px)) \, x^{2q-1} \, dx = \frac{\Gamma\left(q + \frac{1}{2}\right)}{2\sqrt{\pi} \, q p^{2q}}. \tag{11.1}$$

The change of variables $t = px$ shows that this formula is equivalent to the special case $p = 1$. This is an instance of *a fake parameter*.

To show that

$$\int_0^\infty (1 - \text{erf}(t)) \, t^{2q-1} \, dt = \frac{\Gamma\left(q + \frac{1}{2}\right)}{2\sqrt{\pi} \, q}, \tag{11.2}$$

integrate by parts, with $u = 1 - \text{erf}\, t$ and $dv = t^{2q-1}$, to obtain

$$\int_0^\infty (1 - \text{erf}(t)) \, t^{2q-1} \, dt = \frac{1}{\sqrt{\pi} \, q} \int_0^\infty t^{2q} e^{-t^2} \, dt. \tag{11.3}$$

The change of variables $s = t^2$ gives the result.

EXAMPLE 11.2. Entry **6.282.1** is

$$\int_0^\infty \text{erf}(qt) e^{-pt} \, dt = \frac{1}{p}\left[1 - \text{erf}\left(\frac{p}{2q}\right)\right] e^{p^2/4q^2}. \tag{11.4}$$

The change of variables $x = qt$ and with $a = p/2q$ converts the entry to

$$\int_0^\infty \text{erf}(x) e^{-2ax} \, dx = \frac{1}{2a}\left[1 - \text{erf}\, a\right] e^{a^2}. \tag{11.5}$$

This follows simply by integrating by parts.

Example 11.3. Entry **6.282.2**, with a minor change from the stated formula in the table, is

$$\int_0^\infty \left[\text{erf}\left(x+\tfrac{1}{2}\right) - \text{erf}\left(\tfrac{1}{2}\right)\right] e^{-\mu x + \frac{1}{4}}\, dx = \frac{1}{\mu}\text{exp}\left(\frac{(\mu+1)^2}{4}\right)\left[1 - \text{erf}\left(\frac{\mu+1}{2}\right)\right].$$

Integration by parts gives

$$\int_0^\infty \left[\text{erf}\left(x+\tfrac{1}{2}\right) - \text{erf}\left(\tfrac{1}{2}\right)\right] e^{-\mu x + \frac{1}{4}}\, dx = \frac{2}{\mu\sqrt{\pi}} \int_0^\infty e^{-x^2-(\mu+1)x}\, dx. \tag{11.6}$$

The result follows by completing the square. This entry in [**4**] has the factor μ in the denominator replaced by $(\mu+1)(\mu+2)$. This is incorrect. The formula stated here is the correct one.

Example 11.4. Entry **6.283.1** states that

$$\int_0^\infty e^{\beta x}\left[1 - \text{erf}(\sqrt{\alpha x})\right]\, dx = \frac{1}{\beta}\left(\frac{\sqrt{\alpha}}{\sqrt{\alpha-\beta}} - 1\right). \tag{11.7}$$

The change of variables $t = \sqrt{\alpha x}$ gives

$$\int_0^\infty e^{\beta x}\left[1 - \text{erf}(\sqrt{\alpha x})\right]\, dx = \frac{2}{\alpha}\int_0^\infty t e^{\beta t^2/\alpha}\text{erfc}\; t\, dt. \tag{11.8}$$

This last integral is $H_{1,1}(-\beta/\alpha)$ and it is evaluated in Example 10.4.

Example 11.5. Entry **6.283.2** states that

$$\int_0^\infty \text{erf}(\sqrt{qt})\, e^{-pt} dt = \frac{\sqrt{q}}{p\sqrt{p+q}}. \tag{11.9}$$

The change of variables $x = \sqrt{qt}$ gives

$$\int_0^\infty \text{erf}(\sqrt{qt})\, e^{-pt} dt = \frac{2}{q}\int_0^\infty x e^{-px^2/q}\text{erf}\; x\, dx. \tag{11.10}$$

Integration by parts, with erf x the term that will be differentiated, gives the result.

Note. The table [**4**] contains many other integrals containing the error function with results involving more advanced special functions. For instance, entry **6.294.1**:

$$\int_0^\infty [1 - \text{erf}(1/x)]\; e^{-\mu^2 x^2}\frac{dx}{x} = -\text{Ei}(-2\mu), \tag{11.11}$$

where `Ei` denotes the *exponential integral*. These will be described in a future publication.

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