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The integrals in Gradshteyn and Ryzhik. Part 26: The exponential integral

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Abstract. The table of Gradshteyn and Ryzhik contains many entries where the evaluation is given in terms of the exponential integral. A selection of these formulas are established.

1. Introduction

The *exponential integral* function is defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\, dt \tag{1.1}$$

for $x < 0$. In the case $x > 0$ we use the Cauchy principal value

$$\mathrm{Ei}(x) = -\lim_{\epsilon \to 0^+} \left[\int_{-x}^{-\epsilon} \frac{e^{-t}}{t}\, dt + \int_{\epsilon}^{\infty} \frac{e^{-t}}{t}\, dt \right]. \tag{1.2}$$

This appears as entry **3.351.6** in [**2**]

Another function defined by an integral is the *logarithmic integral*:

$$\mathrm{li}(u) := \int_0^u \frac{dx}{\ln x}. \tag{1.3}$$

This is entry **4.211.2**. The change of variables $t = \ln x$ shows that

$$\mathrm{li}(u) = \mathrm{Ei}(\ln u). \tag{1.4}$$

Observe that the integral defining li diverges as $u \to \infty$. Indeed, entry **4.211.1** states that

$$\int_e^{\infty} \frac{dx}{\ln x} = +\infty \tag{1.5}$$

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This is evident from the change of variables $t = \ln x$ that yields

$$\int_e^\infty \frac{dx}{\ln x} = \int_1^\infty \frac{e^t\,dt}{t} \geqslant \int_1^\infty \frac{dt}{t} = \infty. \tag{1.6}$$

2. Some simple changes of variables

The change of variables $t = -as$ yields

$$\int_{-x/a}^\infty \frac{e^{-as}}{s} ds = -\mathrm{Ei}(x). \tag{2.1}$$

Replacing x by ax, this gives

$$\int_{-ax}^\infty \frac{e^{-t}}{t} dt = -\mathrm{Ei}(ax). \tag{2.2}$$

The special choice $x = -a$ in (2.1) yields entry **3.351.5**:

$$\int_1^\infty \frac{e^{-as}}{s} ds = -\mathrm{Ei}(-a). \tag{2.3}$$

The expression

$$\mathrm{Ei}(-a) = -\int_1^\infty \frac{e^{-as}}{s}\, ds \tag{2.4}$$

is an analytic function of a for $\mathrm{Re}\, a > 0$. This provides an analytic extension of $\mathrm{Ei}(z)$ to the left half plane $\mathrm{Re}\, z < 0$. Several entries of [**2**] are derived from here.

Example 2.1. For any β such that $u + \beta > 0$

$$\mathrm{Ei}(-au - a\beta) = \mathrm{Ei}(-a(u+\beta)) = -\int_{u+\beta}^\infty \frac{e^{-ax}}{x} dx \tag{2.5}$$

and then the shift $x \mapsto x + \beta$ produces

$$\mathrm{Ei}(-au - a\beta) = -e^{-a\beta} \int_u^\infty \frac{e^{-ax}}{x+\beta}\, dx \tag{2.6}$$

that can be written as

$$\int_u^\infty \frac{e^{-ax}}{x+\beta}\, dx = -e^{a\beta}\mathrm{Ei}(-au - a\beta). \tag{2.7}$$

This appears as entry **3.352.2**. This representation is valid or $\beta \in \mathbb{C}$ outside the half-line $(-\infty, u]$.

Example 2.2. The special case $u = 0$ and $\beta \notin (-\infty, 0]$ gives

$$\int_0^\infty \frac{e^{-ax}}{x+\beta}\, dx = -e^{a\beta}\mathrm{Ei}(-a\beta). \tag{2.8}$$

This is entry **3.352.4** in [**2**].

Example 2.3. The difference of (2.7) and (2.8) produces

$$\int_0^u \frac{e^{-ax}}{x+\beta}\,dx = e^{au}\left[\mathrm{Ei}(-au-a\beta) - \mathrm{Ei}(-a\beta)\right]. \tag{2.9}$$

This is entry **3.352.1**.

Example 2.4. Entry **3.352.3** states that

$$\int_u^v \frac{e^{-ax}}{x+\beta}\,dx = e^{a\beta}\left[\mathrm{Ei}(-a(v+\beta)) - \mathrm{Ei}(-a(u+\beta))\right]. \tag{2.10}$$

This comes directly from (2.7):

$$\begin{aligned} \int_u^v \frac{e^{-ax}\,dx}{x+\beta} &= \int_u^\infty \frac{e^{-ax}\,dx}{x+\beta} - \int_v^\infty \frac{e^{-ax}\,dx}{x+\beta} \\ &= -e^{a\beta}\mathrm{Ei}(-au-a\beta) + e^{a\beta}\mathrm{Ei}(-av-a\beta). \end{aligned} \tag{2.11}$$

This is the result.

Example 2.5. In the expression (2.7), when $u > 0$, the parameter β may be taken in the range $\beta < u$, so that $x - \beta > 0$ for all $x \geqslant u$. This produces entry **3.352.5**

$$\int_u^\infty \frac{e^{-ax}\,dx}{x-\beta} = -e^{-a\beta}\mathrm{Ei}(-a(u-\beta)). \tag{2.12}$$

Example 2.6. In the case $u = 0$ and $\beta < 0$, the entry in Example 2.5 can be written as

$$\int_0^\infty \frac{e^{-ax}\,dx}{\beta - x} = e^{-a\beta}\mathrm{Ei}(a\beta). \tag{2.13}$$

This is entry **3.352.6** in [**2**].

3. Entries obtained by differentiation

This section presents proofs of some entries in [**2**] obtained by manipulations of derivatives of the exponential integral function.

Example 3.1. Entry **3.353.3** is

$$\int_0^\infty \frac{e^{-ax}\,dx}{(x+\beta)^2} = \frac{1}{\beta} + ae^{-a\beta}\mathrm{Ei}(-a\beta). \tag{3.1}$$

To establish this, differentiate (2.7) and use

$$\frac{d}{dt}\mathrm{Ei}(u) = \frac{e^u}{u}\frac{du}{dt} \tag{3.2}$$

to obtain

$$\int_u^\infty \frac{e^{-ax}\,dx}{(x+\beta)^2} = \frac{e^{-au}}{u+\beta} + ae^{a\beta}\mathrm{Ei}(-au-a\beta). \tag{3.3}$$

The choice $u = 0$ with $\mathrm{Re}\,\beta > 0$ and $\mathrm{Re}\,a > 0$ gives the result.

Example 3.2. Entry **3.353.1** states that

$$\int_u^\infty \frac{e^{-ax}\,dx}{(x+\beta)^n} = e^{-au}\sum_{k=1}^{n-1}\frac{(k-1)!(-a)^{n-k-1}}{(n-1)!(u+\beta)^k} - \frac{(-a)^{n-1}}{(n-1)!}e^{a\beta}\text{Ei}(-au-a\beta). \tag{3.4}$$

can be easily established by induction. The initial step $n=2$ is (3.3). Simply differentiate (3.4) with respect to β to move from n to $n+1$. The details are left to the reader.

Example 3.3. The special case $u=0$ of (3.4) gives

$$\int_0^\infty \frac{e^{-ax}\,dx}{(x+\beta)^n} = \sum_{k=1}^{n-1}\frac{(k-1)!(-a)^{n-k-1}}{(n-1)!\beta^k} - \frac{(-a)^{n-1}}{(n-1)!}e^{a\beta}\text{Ei}(-a\beta). \tag{3.5}$$

This is entry **3.353.2** in [**2**].

Example 3.4. Entry **3.351.4** states that

$$\int_u^\infty \frac{e^{-ax}\,dx}{x^{n+1}} = e^{-au}\sum_{k=1}^{n}\frac{(k-1)!(-a)^{n-k}}{n!u^k} + (-1)^{n+1}\frac{a^n}{n!}\text{Ei}(-au). \tag{3.6}$$

This results follows directly from (3.4) by taking $\beta=0$ and $u>0$ and then replacing n by $n+1$. Changing the index of summation $k\mapsto n-k$, this may be written as it appears in [**2**]

$$\int_u^\infty \frac{e^{-ax}\,dx}{x^{n+1}} = \frac{e^{-au}}{u^n}\sum_{k=1}^{n}\frac{(-1)^k a^k u^k}{n(n-1)\cdots(n-k)} + (-1)^{n+1}\frac{a^n}{n!}\text{Ei}(-au). \tag{3.7}$$

Example 3.5. Entry **3.353.5** states that

$$\int_0^\infty \frac{x^n e^{-ax}}{x+\beta}\,dx = (-1)^{n-1}\beta^n e^{a\beta}\text{Ei}(-a\beta) + \sum_{k=1}^{n}(k-1)!(-\beta)^{n-k}\mu^{-k}. \tag{3.8}$$

In the special case $n=1$, this reduces to

$$\int_0^\infty \frac{xe^{-ax}}{x+\beta}\,dx = \beta e^{a\beta}\text{Ei}(-a\beta) + \frac{1}{a} \tag{3.9}$$

which follows by differentiating (2.8) with respect to a. The general formula (3.8) is obtained directly by further differentiation.

Note 3.6. The entry **3.353.4**

$$\int_0^1 \frac{xe^x\,dx}{(x+1)^2} = \frac{e}{2}-1, \tag{3.10}$$

which does not involve the exponential integral function, can be evaluated by simply integration by parts. This entry has been included in Section 10 of [**1**].

4. Entries with quadratic denominators

This section considers the entries in [**2**] where the integrand is an exponential term divided by a quadratic polynomial.

Example 4.1. Entry **3.354.3** is

$$\int_0^\infty \frac{e^{-ax}\,dx}{\beta^2 - x^2} = \frac{1}{2\beta}\left[e^{-a\beta}\text{Ei}(a\beta) - e^{a\beta}\text{Ei}(-a\beta)\right]. \tag{4.1}$$

To evaluate this integral, assume $\beta \notin \mathbb{R}$ and use the partial fraction decomposition

$$\frac{1}{\beta^2 - x^2} = \frac{1}{2\beta}\left(\frac{1}{\beta - x} - \frac{1}{\beta + x}\right) \tag{4.2}$$

to obtain

$$\int_0^\infty \frac{e^{-ax}\,dx}{\beta^2 - x^2} = \frac{1}{2\beta}\left(\int_0^\infty \frac{e^{-ax}\,dx}{\beta - x} + \int_0^\infty \frac{e^{-ax}\,dx}{\beta + x}\right) \tag{4.3}$$

and now the result comes from (2.8) and (2.13). For $\beta \in \mathbb{R}$ the results valid as a Cauchy principal value integral.

Example 4.2. Differentiate (4.1) with respect to a produces

$$\int_0^\infty \frac{xe^{-ax}\,dx}{\beta^2 - x^2} = \frac{1}{2}\left[e^{-a\beta}\text{Ei}(a\beta) - e^{a\beta}\text{Ei}(-a\beta)\right]. \tag{4.4}$$

This appears as entry **3.354.4** in [**2**].

Example 4.3. Entry **3.354.1**

$$\int_0^\infty \frac{e^{-ax}\,dx}{\beta^2 + x^2} = \frac{1}{\beta}\left[\text{ci}(a\beta)\sin a\beta - \text{si}(a\beta)\cos a\beta\right] \tag{4.5}$$

involves the cosine and sine integrals defined by

$$\text{ci}(u) = -\int_u^\infty \frac{\cos t}{t}\,dt \text{ and } \text{si}(u) = -\int_u^\infty \frac{\sin t}{t}\,dt. \tag{4.6}$$

Start by replacing β by $i\beta$ in (4.1) to obtain

$$\int_0^\infty \frac{e^{-ax}\,dx}{\beta^2 + x^2} = \frac{1}{2i\beta}\left[e^{ia\beta}\text{Ei}(-ia\beta) - e^{-ia\beta}\text{Ei}(ia\beta)\right]. \tag{4.7}$$

The classical identity of Euler

$$e^{\pm i\beta} = \cos a\beta \pm i\sin a\beta \tag{4.8}$$

gives the relation

$$\text{Ei}(\pm ia\beta) = \text{ci}(a\beta) \pm i\,\text{si}(a\beta). \tag{4.9}$$

Replacing in (4.7) gives the result.

Example 4.4. Differentiation of the entry in Example 4.3 gives

$$\int_0^\infty \frac{xe^{-ax}\,dx}{\beta^2 + x^2} = -\text{ci}(a\beta)\sin a\beta - \text{si}(a\beta)\cos a\beta. \tag{4.10}$$

This is entry **3.354.2** in [**2**].

The entries in Sections **3.355** and **3.356** are obtained by differentiation of the entries in Section **3.354** given above.

Example 4.5. Entry **3.355.1** is

$$\int_0^\infty \frac{e^{-ax}\,dx}{(\beta^2+x^2)^2} = \frac{1}{2\beta^2}\left\{\operatorname{ci}(a\beta)\sin(a\beta)-\operatorname{si}(a\beta)\cos(a\beta)- a\beta\left[\operatorname{ci}(a\beta)\cos(a\beta)+\operatorname{si}(a\beta)\sin(a\beta)\right]\right\}. \tag{4.11}$$

This is obtained by differentiation of Entry **3.354.1** given in (4.5).

Example 4.6. Entry **3.355.2** is

$$\int_0^\infty \frac{xe^{-ax}\,dx}{(\beta^2+x^2)^2} = \frac{1}{2\beta^2}\left[1-a\beta\left(\operatorname{ci}(a\beta)\sin(a\beta)-\operatorname{si}(a\beta)\cos(a\beta)\right)\right]. \tag{4.12}$$

This entry appeared with a typo in [**2**]. This entry is obtained by direct differentiation of (4.11).

Example 4.7. Differentiation of entries **3.354.3** and **3.354.4** produce

$$\int_0^\infty \frac{e^{-ax}\,dx}{(\beta^2-x^2)^2} = \frac{1}{4\beta^3}\left[(a\beta-1)e^{a\beta}\operatorname{Ei}(-a\beta)+(1+a\beta)e^{-a\beta}\operatorname{Ei}(a\beta)\right] \tag{4.13}$$

and

$$\int_0^\infty \frac{xe^{-ax}\,dx}{(\beta^2-x^2)^2} = \frac{1}{4\beta^2}\left[-2+a\beta\left(e^{-a\beta}\operatorname{Ei}(a\beta)-e^{a\beta}\operatorname{Ei}(-a\beta)\right]\right). \tag{4.14}$$

These are entries **3.355.3** and **3.355.4**, respectively.

Example 4.8. Differentiating (4.5) $2n$-times with respect to a, gives

$$\int_0^\infty \frac{x^{2n}e^{-ax}\,dx}{\beta^2+x^2} = (-1)^{n-1}\beta^{2n}\left[\operatorname{ci}(a\beta)\cos(a\beta)+\operatorname{si}(a\beta)\sin(a\beta)\right]+ +\frac{1}{\beta^{2n}}\sum_{k=1}^{n}(2n-2k+1)!(-a^2\beta^2)^{k-1}. \tag{4.15}$$

This appears as Entry **3.356.2**. The identity

$$\int_0^\infty \frac{x^{2n}e^{-ax}\,dx}{\beta^2-x^2} = \frac{1}{2}\beta^{2n-1}\left[e^{-a\beta}\operatorname{Ei}(a\beta)-e^{a\beta}\operatorname{Ei}(-a\beta)\right] -\frac{1}{\beta^{2n-1}}\sum_{k=1}^{n}(2n-2k)!(a^2\beta^2)^{k-1} \tag{4.16}$$

is obtained by differentiating (4.1). This appears as Entry **3.356.4**.

Example 4.9. The entries **3.356.1**

$$\int_0^\infty \frac{x^{2n+1}e^{-ax}\,dx}{\beta^2+x^2} = (-1)^{n-1}\beta^{2n}\left[\operatorname{ci}(a\beta)\cos a\beta+\operatorname{si}(a\beta)\sin a\beta\right] +\frac{1}{a^{2n}}\sum_{k=1}^{n}(2n-2k+1)!(-a^2\beta^2)^{k-1} \tag{4.17}$$

and entry **3.356.3**

$$\int_0^\infty \frac{x^{2n+1}e^{-ax}\,dx}{\beta^2 - x^2} = \frac{1}{2}\beta^{2n}\left[e^{a\beta}\text{Ei}(-a\beta) + e^{-a\beta}\text{Ei}(a\beta)\right] - \frac{1}{a^{2n}}\sum_{k=1}^{n}(2n-2k+1)!(a^2\beta^2)^{k-1} \tag{4.18}$$

are obtained by differentiating the entries in Example 4.8.

5. Some higher degree denominators

This section evaluates a series of entries in [**2**] where the integrand is an exponential times a rational function with denominator of degree larger than 2.

Example 5.1. Entry **3.358.1** is

$$\int_0^\infty \frac{e^{-ax}\,dx}{\beta^4 - x^4} = \frac{1}{4\beta^3}\left\{e^{-a\beta}\text{Ei}(a\beta) - e^{a\beta}\text{Ei}(-a\beta) + 2\,\text{ci}(a\beta)\sin(a\beta) - 2\,\text{si}(a\beta)\cos(a\beta)\right\} \tag{5.1}$$

Start with the partial fraction decomposition

$$\frac{1}{\beta^4 - x^4} = \frac{1}{2\beta^2}\left(\frac{1}{\beta^2 - x^2} + \frac{1}{\beta^2 + x^2}\right) \tag{5.2}$$

which shows that the integral in question is a combination of (4.1) and (4.5). The result follows from here.

Example 5.2. Entry **3.358.2**

$$\int_0^\infty \frac{xe^{-ax}\,dx}{\beta^4 - x^4} = \frac{1}{4\beta^2}\left\{e^{a\beta}\text{Ei}(-a\beta) + e^{-a\beta}\text{Ei}(a\beta) - 2\,\text{ci}(a\beta)\cos(a\beta) - 2\,\text{si}(a\beta)\sin(a\beta)\right\}. \tag{5.3}$$

This is obtained by differentiation of (5.1). The entries **3.358.3**

$$\int_0^\infty \frac{x^2e^{-ax}\,dx}{\beta^4 - x^4} = \frac{1}{4\beta}\left\{e^{-a\beta}\text{Ei}(a\beta) - e^{a\beta}\text{Ei}(-a\beta) - 2\,\text{ci}(a\beta)\sin(a\beta) + 2\,\text{si}(a\beta)\cos(a\beta)\right\} \tag{5.4}$$

and **3.358.4**

$$\int_0^\infty \frac{x^3e^{-ax}\,dx}{\beta^4 - x^4} = \frac{1}{4}\left\{e^{a\beta}\text{Ei}(-a\beta) + e^{-a\beta}\text{Ei}(a\beta) + 2\,\text{ci}(a\beta)\cos(a\beta) + 2\,\text{si}(a\beta)\sin(a\beta)\right\} \tag{5.5}$$

come from further differentiation.

The entries in Section **3.357** can be established by algebraic manipulations of the examples given above.

Example 5.3. Entry **3.357.1** states that

$$\int_0^\infty \frac{e^{-ax}\,dx}{\beta^3+\beta^2x+\beta x^2+x^3} = \frac{1}{2\beta^2}\left\{ \mathrm{ci}(a\beta)(\sin a\beta+\cos(a\beta))+ \mathrm{si}(a\beta)(\sin a\beta-\cos(a\beta))-e^{a\beta}\mathrm{Ei}(-a\beta)\right\} \tag{5.6}$$

This formula is obtained from (5.1) and (5.3) and the algebraic identity

$$\frac{1}{\beta^3+\beta^2x+\beta x^2+x^3} = \frac{\beta-x}{\beta^4-x^4}. \tag{5.7}$$

Example 5.4. Differentiation of (5.6) gives

$$\int_0^\infty \frac{xe^{-ax}\,dx}{\beta^3+\beta^2x+\beta x^2+x^3} = \frac{1}{2\beta}\left\{ \mathrm{ci}(a\beta)(\sin a\beta-\cos(a\beta)) -\mathrm{si}(a\beta)(\sin a\beta+\cos(a\beta))-e^{a\beta}\mathrm{Ei}(-a\beta)\right\} \tag{5.8}$$

This is entry **3.357.2** in [**2**].

Example 5.5. Differentiating (5.8) produces entry **3.357.3**:

$$\int_0^\infty \frac{x^2e^{-ax}\,dx}{\beta^3+\beta^2x+\beta x^2+x^3} = \frac{1}{2}\left\{ -\mathrm{ci}(a\beta)(\sin a\beta+\cos(a\beta)) -\mathrm{si}(a\beta)(\sin a\beta-\cos(a\beta))-e^{a\beta}\mathrm{Ei}(-a\beta)\right\}. \tag{5.9}$$

The identity

$$\frac{1}{\beta^3-\beta^2x+\beta x^2-x^3} = \frac{\beta+x}{\beta^4-x^4} \tag{5.10}$$

and the method used to establish the last three entries produces proofs of the next three.

Example 5.6. Entry **3.357.4** is

$$\int_0^\infty \frac{e^{-ax}\,dx}{\beta^3-\beta^2x+\beta x^2-x^3} = \frac{1}{2\beta^2}\left\{ \mathrm{ci}(a\beta)(\sin a\beta-\cos(a\beta)) -\mathrm{si}(a\beta)(\sin a\beta+\cos(a\beta))+e^{-a\beta}\mathrm{Ei}(a\beta)\right\} \tag{5.11}$$

and **3.357.5** is

$$\int_0^\infty \frac{xe^{-ax}\,dx}{\beta^3-\beta^2x+\beta x^2-x^3} = \frac{1}{2\beta}\left\{ -\mathrm{ci}(a\beta)(\sin a\beta+\cos(a\beta)) -\mathrm{si}(a\beta)(\sin a\beta-\cos(a\beta))+e^{-a\beta}\mathrm{Ei}(a\beta)\right\} \tag{5.12}$$

and, finally, entry **3.357.6** is

$$\int_0^\infty \frac{x^2e^{-ax}\,dx}{\beta^3-\beta^2x+\beta x^2-x^3} = \frac{1}{2}\left\{ \mathrm{ci}(a\beta)(\cos a\beta-\sin(a\beta)) +\mathrm{si}(a\beta)(\cos a\beta+\sin(a\beta))+e^{-a\beta}\mathrm{Ei}(a\beta)\right\}. \tag{5.13}$$

6. Entries involving absolute values

This section presents the evaluation of some entries in [**2**] where the integrand contains variations of the function $\ln |x|$.

Example 6.1. Entry **4.337.3** states that

$$\int_0^\infty e^{-\mu x} \ln |a-x| \, dx = \frac{1}{\mu} \left[\ln a - e^{-a\mu} \mathrm{Ei}(a\mu) \right]. \tag{6.1}$$

To establish this entry observe that the singularity at $x=a$ is integrable and that

$$\frac{d}{dx} \ln |a-x| = \frac{1}{a-x}. \tag{6.2}$$

Integration by parts produces

$$\begin{aligned} \int_0^\infty e^{-\mu x} \ln |a-x| \, dx &= -\frac{1}{\mu} \int_0^\infty \ln |x-a| de^{-\mu x} \\ &= -\frac{1}{\mu} \left(-\log a - e^{-\mu a} \int_0^\infty \frac{e^{-\mu x}}{x-a} \, dx \right) \\ &= \frac{1}{\mu} \left(\ln a + e^{-\mu t} \int_{-\mu a}^\infty \frac{e^{-u}}{u} \, du \right) \\ &= \frac{1}{\mu} \left(\ln a - e^{-\mu a} \mathrm{Ei}(\mu a) \right). \end{aligned}$$

This is the result.

Example 6.2. Entry **4.337.4** states that

$$\int_0^\infty e^{-\mu x} \ln \left| \frac{\beta}{\beta - x} \right| \, dx = \frac{1}{\mu} e^{-\beta\mu} \mathrm{Ei}(\beta\mu). \tag{6.3}$$

This evaluation is obtained directly from (6.1) and the identity

$$\int_0^\infty e^{-\mu x} \ln \left| \frac{\beta}{\beta - x} \right| \, dx = \ln |\beta| \int_0^\infty e^{-\mu x} \, dx - \int_0^\infty e^{-\mu x} \ln |\beta - x| \, dx. \tag{6.4}$$

7. Some integrals involving the logarithm function

The exponential integral function Ei allows the evaluation of a variety of entries in [**2**] containing a logarithmic term. For instance **4.212.1**:

$$\int_0^1 \frac{dx}{a + \ln x} = e^{-a} \mathrm{Ei}(a) \tag{7.1}$$

follows from the change of variables $t = a + \ln x$. Similarly, **4.212.2**:

$$\int_0^1 \frac{dx}{a - \ln x} = -e^{a} \mathrm{Ei}(-a) \tag{7.2}$$

is evaluated using $t = a - \ln x$.

We now consider the family

$$f_n(a) := \int_0^1 \frac{dx}{(a+\ln x)^n}. \tag{7.3}$$

The change of variables $t = a + \ln x$ gives

$$f_n(a) = e^{-a} \int_{-\infty}^a t^{-n} e^t \, dt. \tag{7.4}$$

Integrate by parts to produce

$$\int_{-\infty}^a \frac{e^t \, dt}{t^n} = \frac{e^a a^{1-n}}{1-n} - \frac{1}{1-n} \int_{-\infty}^a \frac{e^t \, dt}{t^{n-1}}. \tag{7.5}$$

This yields a recurrence for the integrals $f_n(a)$:

$$f_n(a) = -\frac{a^{1-n}}{n-1} + \frac{1}{n-1} f_{n-1}(a). \tag{7.6}$$

The initial value is given in **4.212.1**. From here we deduce and prove by induction, formula **4.212.8**:

$$\int_0^1 \frac{dx}{(a+\ln x)^n} = \frac{e^{-a}}{(n-1)!}\mathrm{Ei}(a) - \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \frac{(n-k-1)!}{a^{n-k}}. \tag{7.7}$$

Using (7.4) we obtain **3.351.4**:

$$\int_a^\infty \frac{e^{-px} \, dx}{x^{n+1}} = \frac{(-1)^{n+1} p^n}{n!}\mathrm{Ei}(-ap) + \frac{e^{-ap}}{a^n n!} \sum_{k=0}^{n-1} (-1)^k p^k a^k (n-k-1)! \tag{7.8}$$

The integral **4.212.3**:

$$\int_0^1 \frac{dx}{(a+\ln x)^2} = -\frac{1}{a} + e^{-a}\mathrm{Ei}(a) \tag{7.9}$$

is the special case $n=2$ of (7.7). The integral **4.212.5**:

$$\int_0^1 \frac{\ln x \, dx}{(a+\ln x)^2} = 1 + (1-a)e^{-a}\mathrm{Ei}(a) \tag{7.10}$$

can be obtained from

$$\frac{\ln x}{(a+\ln x)^2} = \frac{1}{a+\ln x} - \frac{a}{(a+\ln x)^2}. \tag{7.11}$$

Similar arguments produce **4.212.9**:

$$\int_0^1 \frac{dx}{(a+\ln x)^n} = \frac{(-1)^n e^a \mathrm{Ei}(-a)}{(n-1)!} + \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^{n-1} (n-k-1)!(-a)^{k-n}. \tag{7.12}$$

The formula **4.212.4**:

$$\int_0^1 \frac{dx}{(a-\ln x)^2} = \frac{1}{a} + e^a \mathrm{Ei}(-a) \tag{7.13}$$

is the special case $n = 2$. Writing

$$\ln x = a - (a - \ln x) \tag{7.14}$$

we obtain the evaluation of **4.212.6**:

$$\int_0^1 \frac{\ln x \, dx}{(a - \ln x)^2} = 1 + (1+a)e^a \text{Ei}(-a). \tag{7.15}$$

8. The exponential scale

Several of the entries in [**2**] contain integrals that can be reduced to the definition of the exponential integral. This section contains some of them.

Example 8.1. Entry **4.331.2** states that

$$\int_1^\infty e^{-\mu x} \ln x \, dx = -\frac{1}{\mu}\text{Ei}(-\mu), \text{ for } \operatorname{Re}\mu > 0. \tag{8.1}$$

To evaluate this entry, assume $\mu > 0$ and integrate by parts to obtain

$$\int_1^\infty e^{-\mu x} \ln x \, dx = \frac{1}{\mu} \int_1^\infty \frac{e^{-\mu x}}{x} \, dx. \tag{8.2}$$

The change of variables $s = -\mu x$ now gives the result for $\mu \in \mathbb{R}$. The case $\mu \in \mathbb{C}$ follows by analytic continuation.

Example 8.2. Entry **4.337.1**

$$\int_0^\infty e^{-\mu x} \ln(\beta + x) \, dx = \frac{1}{\mu}\left[\ln\beta - e^{\mu\beta}\text{Ei}(-\beta\mu)\right], \text{ for } |\arg\beta| < \pi, \operatorname{Re}\mu > 0 \tag{8.3}$$

can be transformed to **4.331.2** by simple changes of variables. Start with $\beta > 0$ and make the change of variables $x = \beta t$ to obtain

$$\int_0^\infty e^{-\mu x} \ln(\beta + x) \, dx = \frac{\ln\beta}{\mu} + \beta \int_0^\infty e^{-\mu\beta t} \ln(1+t) \, dt. \tag{8.4}$$

The change of variables $s = t + 1$ and Entry **4.331.2** gives the result.

Example 8.3. Entry **4.337.2** is

$$\int_0^\infty e^{-\mu x} \ln(1 + \beta x) \, dx = -\frac{1}{\mu} e^{\mu/\beta}\text{Ei}(-\mu/\beta). \tag{8.5}$$

The change of variables $t = \beta x$ reduces this integral to **4.337.1** with $\mu \mapsto \mu/\beta$ and $\beta \mapsto 1$.

The change of variables $t = -ae^{nu}$ produces

$$\text{Ei}(x) = -n \int_c^\infty \exp\left(-ae^{nu}\right) \, du, \tag{8.6}$$

where $c = \frac{1}{n}\ln(-x/a)$. The choice $x = -a$ produces

$$\text{Ei}(-a) = -n \int_0^\infty \exp\left(-ae^{nu}\right). \tag{8.7}$$

This appears as **3.327** in [**2**].

Some further examples of entries in [**2**], containing the exponential integral function, will be described in a future publication.

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