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The integrals in Gradshteyn and Ryzhik. Part 28: The confluent hypergeometric function and Whittaker functions

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Abstract. The table of Gradshteyn and Ryzhik contains many entries where the integrand contains the confluent hypergeometric function ${}_1F_1(a;c;z)$ and a closely associated function studied by Whittaker. A selection of these entries are evaluated.

1. Introduction

The confluent hypergeometric function, denoted by ${}_1F_1(a;c;z)$, is defined by

$$ {}_1F_1(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!} \tag{1.1} $$

with $(a)_n$ being the rising factorial

$$ (a)_n := a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, $$

for $a \in \mathbb{C}$. It arises when two of the regular singular points of the differential equation for the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$, given by

$$ z(1-z)y'' + (c-(a+b+1)z)y' - aby = 0, \tag{1.2} $$

are allowed to merge into one singular point. More specifically, if we replace z by z/b in ${}_2F_1(a,b;c;z)$, then the corresponding differential equation has singular points at 0, b and ∞. Now let $b \to \infty$ so as to have infinity as a confluence of two singularities. This results in the function ${}_1F_1(a;c;z)$ so that

$$ {}_1F_1(a;c;z) = \lim_{b\to\infty} {}_2F_1\left(a,b;c;\frac{z}{b}\right), \tag{1.3} $$

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and the corresponding differential equation

$$zy'' + (c - z)y' - ay = 0, \tag{1.4}$$

known as the confluent hypergeometric equation. Following two transformation formulas for ${}_1F_1$, due to Kummer, are very useful:

$$\begin{aligned} {}_1F_1(a;c;z) &= e^z {}_1F_1(c-a;c;-z) \qquad (b \neq 0, -1, -2, \cdots), \\ {}_1F_1(a;2a;2z) &= e^z {}_0F_1\left(-; a+\frac{1}{2}; \frac{z^2}{4}\right) \quad (2a \text{ is not an odd integer} < 0). \end{aligned} \tag{1.5}$$

The confluent hypergeometric function has many different notations other than ${}_1F_1(a;c;z)$, for example, $\Phi(a;c;z)$ [**2**, p. 1023] or $M(a;c;z)$ [**4**]. Closely associated to ${}_1F_1(a;c;z)$ are the Whittaker functions $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ defined by [**2**, p. 1024]

$$M_{k,\mu}(z) = z^{\mu+\frac{1}{2}} e^{-z/2} {}_1F_1\left(\mu - k + \frac{1}{2}; 2\mu+1; z\right), \tag{1.6}$$

$$W_{k,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)} M_{k,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-k)} M_{k,-\mu}(z). \tag{1.7}$$

In this paper, the formulas in Sections 7.612 and 7.621 of [**2**] are established. These involve the confluent hypergeometric function and the Whittaker functions. The remaining entries involving these functions will be considered in the future.

The asymptotic formulas for these functions have been well-studied in the literature. Some are collected here for the benefit of the reader. The first one is an asymptotic expansion for ${}_1F_1$

$${}_1F_1(a;c;z) \sim \frac{\Gamma(c) e^z z^{a-c}}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(c-a)_n (1-a)_n}{n!} z^{-n} \quad \text{as } z \to \infty, \tag{1.8}$$

for $|\arg(z)| < \frac{\pi}{2}$. This appears in [**4**, p. 174, equation (7.9)]. The more general asymptotic expansion

$$\begin{aligned} {}_1F_1(a;c;z) \sim\ & e^z z^{a-c} \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(c-a)_n (1-a)_n}{n!} z^{-n} \\ & + \frac{\Gamma(c)\, e^{\pm i\pi a}}{\Gamma(c-a)} z^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (1+a-c)_n}{n!} (-z)^{-n}, \end{aligned} \tag{1.9}$$

where the upper sign is taken of $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$ and the lower sign in the case $-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi$. The first part dominates the second when $\mathrm{Re}\,(z) > 0$ corresponding with (1.8). The second part is dominant for $\mathrm{Re}\,(z) < 0$. This appears in [**4**, p. 189, Exercise (7.7)].

The Ψ-function is defined in [**2**, p. 1023] by

$$\Psi(a;c;z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} {}_1F_1(a;c;z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c} {}_1F_1(a-c+1; 2-c; z) \tag{1.10}$$

and its asymptotic behavior is given by

$$\Psi(a;c;z) \sim z^{-a}\sum_{n=0}^{\infty}\frac{(a)_n(a-c+1)_n}{n!}(-z)^{-n}, \text{ as } z\to\infty \text{ for } |\arg z|<\tfrac{3}{2}\pi, \tag{1.11}$$

(see [**4**, p. 175, formula (7.13)]).

The first formula established here is Entry **7.612.1**. This is a standard result for the Mellin transform of ${}_1F_1(a;c;-t)$ [**1**, p. 192]. A proof is presented here to make the results self-contained. The argument begins with an entry in [**2**].
Entry **7.612.1** states that for $0<\mathrm{Re}\,(b)<\mathrm{Re}\,(a)$,

$$\int_0^\infty t^{b-1}{}_1F_1(a;c;-t)\,dt = \frac{\Gamma(b)\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}. \tag{1.12}$$

Proof. We need $\mathrm{Re}\,(b)>0$ for the convergence of the integral near $t=0$. Since the argument of ${}_1F_1(a;c;-t)$ is in the left half-plane, the second expression in the asymptotic expansion in (1.9) becomes dominant. Hence we require $\mathrm{Re}\,(b)<\mathrm{Re}\,(a)$ for the convergence of the integral near ∞.

Apply Kummer's first transformation in (1.5) on the left-hand side of (1.12) so that

$$\begin{aligned}\int_0^\infty t^{b-1}{}_1F_1(a;c;-t)\,dt &= \int_0^\infty t^{b-1}e^{-t}{}_1F_1(c-a;c;t)\,dt\\ &= \int_0^\infty t^{b-1}e^{-t}\sum_{n=0}^{\infty}\frac{(c-a)_nt^n}{(c)_nn!}\,dt\\ &= \sum_{n=0}^{\infty}\frac{(c-a)_n}{(c)_nn!}\int_0^\infty t^{b+n-1}e^{-t}\,dt\\ &= \sum_{n=0}^{\infty}\frac{(c-a)_n\Gamma(b+n)}{(c)_nn!}\\ &= \Gamma(b)\,{}_2F_1(c-a,b;c;1)\\ &= \frac{\Gamma(b)\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)},\end{aligned}$$

where the last equality follows from Gauss' formula for evaluating ${}_2F_1$ at 1 [**1**, p. 66, Theorem 2.2.2].

The next evaluation is Entry **7.612.2**. It states that, for $0<\mathrm{Re}\,(b)<\mathrm{Re}\,(a)$ and $\mathrm{Re}\,(c)<\mathrm{Re}\,(b+1)$,

$$\int_0^\infty t^{b-1}\Psi(a;c;t)\,dt = \frac{\Gamma(b)\Gamma(a-b)\Gamma(b-c+1)}{\Gamma(a)\Gamma(a-c+1)}, \tag{1.13}$$

where the function $\Psi(a;c;t)$ is defined in (1.10).

Proof. Use (1.10) in the integrand and write the given integral as the sum of two integrals. The condition $\mathrm{Re}\,(b)>0$ is required for the convergence of the first integral near $t=0$ where as the the second integral requires $\mathrm{Re}\,(b-c+1)>0$. The behavior of the integral in (1.13) near ∞ requires $\mathrm{Re}\,(b-a)<0$ as can be seen by using (1.11).

The following integral representation for $\Psi(a;c;z)$, valid for $\text{Re}\,(a) > 0$ and $\text{Re}\,(z) > 0$, is used in the argument:

$$\Psi(a;c;z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{c-a-1}\, dt. \tag{1.14}$$

This is Entry **9.211.4** in [**2**, p. 1023]. A proof appears in [**4**, p. 174-175].

Using (1.14) on the left-hand side of (1.13), it follows that

$$\int_0^\infty t^{b-1}\Psi(a;c;t)\,dt = \frac{1}{\Gamma(a)} \int_0^\infty t^{b-1}\,dt \int_0^\infty e^{-tx} x^{a-1}(1+x)^{c-a-1}\,dx. \tag{1.15}$$

Interchanging the order of integration, one obtains

$$\begin{aligned}
\int_0^\infty t^{b-1}\Psi(a;c;t)\,dt &= \frac{1}{\Gamma(a)} \int_0^\infty x^{a-1}(1+x)^{c-a-1}\,dx \int_0^\infty e^{-tx} t^{b-1}\,dt \\
&= \frac{\Gamma(b)}{\Gamma(a)} \int_0^\infty x^{a-b-1}(1+x)^{c-a-1}\,dx \\
&= \frac{\Gamma(b) B(a-b, b-c+1)}{\Gamma(a)} \\
&= \frac{\Gamma(b)\Gamma(a-b)\Gamma(b-c+1)}{\Gamma(a)\Gamma(a-c+1)},
\end{aligned}$$

where the last two steps follow from the classical representation for Euler's beta function $B(x,y)$:

$$B(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}\,dx \quad \text{for } \text{Re}\,(a) > 0, \text{Re}\,(b) > 0 \tag{1.16}$$

and its expression in terms of the gamma function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{1.17}$$

This completes the proof.

2. A sample of formulas

This section collects a selection of formulas from [**2**] involving the confluent hypergeometric function. The first example is **7.621.4**.

Example 2.1. Entry **7.621.4** states

$$\int_0^\infty e^{-st} t^{b-1} {}_1F_1(a;c;kt)\,dt = \begin{cases} \Gamma(b) s^{-b} {}_2F_1\left(\begin{matrix} a \quad b \\ c \end{matrix} \Bigg| \frac{k}{s} \right) & \text{if } |s| > |k| \\ \Gamma(b)(s-k)^{-b} {}_2F_1\left(\begin{matrix} c-a \quad b \\ c \end{matrix} \Bigg| \frac{k}{k-s} \right) & \text{if } |s-k| > |k|, \end{cases} \tag{2.1}$$

where $\text{Re}\,(b) > 0$ and $\text{Re}\,(s) > \max\{0, \text{Re}\,(k)\}$.

Proof. Assume first $\text{Re}\,(k) > 0$. Using (1.8), it follows that, as $t \to \infty$, the integrand behaves like $e^{(k-s)t}$, and in order to ensure convergence, the condition $\text{Re}\,(k-s) < 0$ is needed. This explains the condition $\text{Re}\,(s) > \text{Re}\,(k)$. As $t \to 0$, the integrand behaves like t^{b-1}. The condition $\text{Re}\,(b) > 0$ is required for the convergence of the integral.

The discussion above guarantees the validity of interchange of summation and integration in the next steps:

$$\begin{aligned}
(2.2)\qquad \int_0^\infty e^{-st}t^{b-1}{}_1F_1(a;c;kt)\,dt &= \sum_{n=0}^\infty \frac{(a)_n k^n}{(c)_n n!}\int_0^\infty e^{-st}t^{b+n-1}\,dt \\
&= \Gamma(b)\sum_{n=0}^\infty \frac{(a)_n}{(c)_n}\frac{k^n}{n!}\frac{\Gamma(b+n)}{\Gamma(b)}\frac{1}{s^{b+n}} \\
&= \frac{\Gamma(b)}{s^b}\sum_{n=0}^\infty \frac{(b)_n(a)_n}{(c)_n n!}\left(\frac{k}{s}\right)^n \\
&= \frac{\Gamma(b)}{s^b}{}_2F_1\left(\begin{matrix} a \quad b \\ c \end{matrix}\middle|\frac{k}{s}\right)
\end{aligned}$$

provided $|k/s| < 1$; i.e., if $|s| > |k|$. This proves the first part.

In the case $\text{Re}\,(k) < 0$ and use Kummer's relation

$$(2.3)\qquad {}_1F_1(a;c;w) = e^w{}_1F_1(c-a;c;-w)$$

(see [**1**, p. 191, equation (4.1.11)]). Therefore, as $t \to \infty$, the integrand behaves like e^{-s} and convergence requires the condition $\text{Re}\,(s) > 0$. Then

$$\begin{aligned}
(2.4)\qquad \int_0^\infty e^{-st}t^{b-1}{}_1F_1(a;c;kt)\,dt &= \int_0^\infty e^{-(s-k)t}t^{b-1}{}_1F_1(c-a;c;-kt)\,dt \\
&= \Gamma(b)\sum_{n=0}^\infty \frac{(c-a)_n}{(c)_n}\frac{\Gamma(b+n)}{\Gamma(b)}\frac{(-k)^n}{(s-k)^{b+n}\,n!} \\
&= \frac{\Gamma(b)}{(s-k)^b}{}_2F_1\left(\begin{matrix} c-a \quad b \\ c \end{matrix}\middle|\frac{k}{k-s}\right).
\end{aligned}$$

Now apply Pfaff's transformation ([**1**, p.68, equation (2.2.6)])

$$(2.5)\qquad {}_2F_1\left(\begin{matrix} a \quad b \\ c \end{matrix}\middle| z\right) = (1-z)^{-b}{}_2F_1\left(\begin{matrix} c-a \quad b \\ c \end{matrix}\middle|\frac{z}{z-1}\right)$$

to the hypergeometric series above to obtain the result. This establishes the first formula when $\text{Re}\,(k) < 0$. The second formula, for the range $|s-k| > |k|$, is established along similar lines.

A direct application of the more general asymptotic expansion (1.9) then reduces the case $\text{Re}\,k = 0$ to the previous two cases (according to the sign of $\text{Im}\,k$).

Example 2.2. Entry **7.621.5** states that

$$(2.6)\qquad \int_0^\infty t^{c-1}{}_1F_1(a;c;t)e^{-st}\,dt = \Gamma(c)s^{-c}(1-s^{-1})^{-a}$$

for $\text{Re}\,(c) > 0$ and $\text{Re}\,(s) > 1$.

Proof. This is actually the special case $k = 1$ and $b = c$ in the first part of **7.621.4**. The condition $|s| > 1$ implies $\text{Re}\,(s) > 1$. Then

$$\begin{aligned}\int_0^\infty t^{c-1}{}_1F_1(a;c;t)e^{-st}\,dt &= \Gamma(c)s^{-c}{}_2F_1\left(\begin{matrix}a & c\\ & c\end{matrix}\middle|\frac{1}{s}\right)\\ &= \Gamma(c)s^{-c}(1-1/s)^{-a}\end{aligned}$$

using

$$\begin{aligned}{}_2F_1\left(\begin{matrix}a & c\\ & c\end{matrix}\middle|u\right) &= \sum_{n=0}^\infty \frac{(a)_n(c)_n}{(c)_n n!}u^n\\ &= (1-u)^{-a}\end{aligned}$$

by the binomial theorem for $|u| < 1$.

Example 2.3. Entry **7.621.6** states that, for $\text{Re}\,(c) < \text{Re}\,(b) + 1$,

$$\int_0^\infty t^{b-1}\Psi(a;c;t)e^{-st}\,dt \tag{2.7}$$
$$= \frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)}\begin{cases}{}_2F_1\left(\begin{matrix}b & b-c+1\\ & a+b-c+1\end{matrix}\middle|1-s\right) & \text{Re}\,(b) > 0, |1-s| < 1,\\ s^{-b}{}_2F_1\left(\begin{matrix}a & b\\ & a+b-c+1\end{matrix}\middle|1-\frac{1}{s}\right) & \text{Re}\,(s) > \frac{1}{2}.\end{cases}$$

Proof. The integral in question is now evaluation in two cases, according to the conditions given in (2.3). The assumptions on the parameters will appear as conditions in the proof.

(i) *First part.* Using the expression for $\Psi(a;c;t)$ in (1.10) gives

$$\begin{aligned}\int_0^\infty t^{b-1}\Psi(a;c;t)e^{-st}\,dt &= \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\int_0^\infty t^{b-1}{}_1F_1(a;c;t)e^{-st}\,dt\\ &+ \frac{\Gamma(c-1)}{\Gamma(a)}\int_0^\infty t^{(b-c+1)-1}{}_1F_1(a-c+1;2-c;t)e^{-st}\,dt\end{aligned} \tag{2.8}$$

The first integral requires $\text{Re}\,(b) > 0$ for convergence near $t = 0$ and the second integral requires $\text{Re}\,(c) < \text{Re}\,(b) + 1$ in order to apply the first formula in **7.621.4**, with $k = 1$. This also requires the condition $|s| > 1$. However, the behavior of the integrand on the left-hand side at infinity renders the integral convergent when $\text{Re}\,(s) > 0$, and as will be seen below, the result holds for $|1 - s| < 1$ by analytic continuation.

A direct application of **7.621.4** gives the value

$$\begin{aligned}\int_0^\infty t^{b-1}\Psi(a;c;t)e^{-st}\,dt &= \frac{\Gamma(1-c)\Gamma(b)s^{-b}}{\Gamma(a-c+1)}{}_2F_1\left(\begin{matrix}a & b\\ & c\end{matrix}\middle|\frac{1}{s}\right)\\ &+ \frac{\Gamma(c-1)\Gamma(b-c+1)}{\Gamma(a)}s^{c-b-1}{}_2F_1\left(\begin{matrix}a-c+1 & b-c+1\\ & 2-c\end{matrix}\middle|\frac{1}{s}\right).\end{aligned} \tag{2.9}$$

The answer is simplified using the identity (see [**4**, p.113, (5.12)]

$$(2.10)\qquad {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(1-z)^{-a}{}_2F_1\left(\begin{matrix} a & c-b \\ a-b+1 & \end{matrix}\middle| \frac{1}{1-z}\right) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(1-z)^{-b}{}_2F_1\left(\begin{matrix} b & c-a \\ b-a+1 & \end{matrix}\middle| \frac{1}{1-z}\right)$$

to produce

$$(2.11)\qquad \int_0^\infty t^{b-1}\Psi(a;c;t)e^{-st}\,dt = \frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)}{}_2F_1\left(\begin{matrix} b & b-c+1 \\ a+b-c+1 & \end{matrix}\middle| 1-s\right),$$

as claimed. Note that the presence of the hypergeometric function on the left of (2.10) requires $|z|<1$; that is, $|1-s|<1$ in this example. The above identity also requires $|\arg(1-z)|<\pi$; that is $|\arg(s)|<\pi$, which is satisfied when $|1-s|<1$.

(ii) *Second part.* The formula (1.10) gives

(2.12)
$$\int_0^\infty t^{b-1}\Psi(a;c;t)e^{-st}\,dt = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\int_0^\infty t^{b-1}{}_1F_1(a;c;t)e^{-st}\,dt + \frac{\Gamma(c-1)}{\Gamma(a)}\int_0^\infty t^{(b-c+1)-1}{}_1F_1(a-c+1;2-c;t)e^{-st}\,dt,$$

and then the second part of **7.621.4** gives

(2.13)
$$\int_0^\infty t^{b-1}\Psi(a;c;t)e^{-st}\,dt = \frac{\Gamma(1-c)\Gamma(b)}{\Gamma(a-c+1)(s-1)^b}{}_2F_1\left(\begin{matrix} c-a & b \\ c & \end{matrix}\middle| \frac{1}{1-s}\right) \quad \frac{\Gamma(c-1)\Gamma(b-c+1)}{\Gamma(a)(s-1)^{b-c+1}}{}_2F_1\left(\begin{matrix} 1-a & b-c+1 \\ 2-c & \end{matrix}\middle| \frac{1}{1-s}\right),$$

which is valid for $|1-s|>1$. To reduce this expression to the form stated in (2.7) one uses the identity

(2.14)
$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}z^{-a}{}_2F_1\left(\begin{matrix} a & a-c+1 \\ a+b-c+1 & \end{matrix}\middle| 1-\frac{1}{z}\right) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}z^{a-c}{}_2F_1\left(\begin{matrix} c-a & 1-a \\ c-a-b+1 & \end{matrix}\middle| 1-\frac{1}{z}\right)$$

valid for $|\arg(1-z)|<\pi$ and $|\arg z|<\pi$. This appears in [**4**, p. 113, (5.13)]. Now take $z=1-1/s$ and replace c by $a+b-c+1$ to obtain

$$(2.15)\qquad {}_2F_1\left(\begin{matrix} a & b \\ a+b-c+1 & \end{matrix}\middle| 1-\frac{1}{s}\right) =$$
$$\frac{\Gamma(a+b-c+1)\Gamma(1-c)}{\Gamma(b-c+1)\Gamma(a-c+1)}\left(1-\frac{1}{s}\right)^{-a}{}_2F_1\left(\begin{matrix} a & c-b \\ c & \end{matrix}\middle| \frac{1}{1-s}\right) + $$
$$+\frac{\Gamma(a+b-c+1)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}\left(1-\frac{1}{s}\right)^{c-b-1}s^{1-c}{}_2F_1\left(\begin{matrix} b-c+1 & 1-a \\ 2-c & \end{matrix}\middle| \frac{1}{1-s}\right)$$

for $|s|>|s-1|$, $|\arg(1/s)|<\pi$ and $|\arg(1-1/s)|<\pi$. Euler's relation

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \Bigg| z \right) = (1-z)^{c-a-b} {}_2F_1\left(\begin{matrix} c-a & c-b \\ & c \end{matrix} \Bigg| z \right) \tag{2.16} $$

on the first hypergeometric function on the right of (2.15) produces

$$ {}_2F_1\left(\begin{matrix} a & c-b \\ & c \end{matrix} \Bigg| \frac{1}{1-s} \right) = \left(1-\frac{1}{1-s}\right)^{b-a} {}_2F_1\left(\begin{matrix} c-a & b \\ & c \end{matrix} \Bigg| \frac{1}{1-s} \right). \tag{2.17} $$

Now use (2.15) and (2.17) to produce the desired result. It is easy to check that the conditions $|s|>|1-s|>1$, and the principle of analytic continuation coupled with the fact that the integral on the left-hand side converges for $\operatorname{Re} s>0$ implies that the result is true for $\operatorname{Re} s>1/2$. Also, the conditions $|\arg(1/s)|<\pi$ and $|\arg(1-1/s)|<\pi$ are satisfied for these values of s.

Example 2.4. Entry **7.621.1** states that

$$ \int_0^\infty e^{-st} t^\alpha M_{\mu,\nu}(t)\, dt = \frac{\Gamma(\alpha+\nu+3/2)}{(s+\frac{1}{2})^{\alpha+\nu+\frac{3}{2}}} {}_2F_1\left(\begin{matrix} \alpha+\nu+\frac{3}{2} & \nu-\mu+\frac{1}{2} \\ & 2\nu+1 \end{matrix} \Bigg| \frac{2}{2s+1} \right). $$

for $\operatorname{Re}\left(\alpha+\mu+\frac{3}{2}\right)>0$ and $\operatorname{Re}(s)>\frac{1}{2}$.

Proof. Note that from (1.6)

$$ M_{\mu,\nu}(t) = t^{\nu+1/2} e^{-t/2} {}_1F_1(\nu-\mu+\tfrac{1}{2}; 2\nu+1; t) \tag{2.18} $$

and this gives

$$ \int_0^\infty e^{-st} t^\alpha M_{\mu,\nu}(t)\, dt = \int_0^\infty e^{-(s+1/2)t} t^{(\alpha+\nu+3/2)-1} {}_1F_1(\nu-\mu+\tfrac{1}{2}; 2\nu+1; t)\, dt. \tag{2.19} $$

Now use the first part of **7.621.4** with $s\mapsto s+\frac{1}{2}$, $b\mapsto\alpha+\nu+\frac{3}{2}$ and $k=1$, $a\mapsto\nu-\mu+\frac{1}{2}$, $c\mapsto 2\nu+1$. This gives the required result. The asymptotics of ${}_1F_1$ as $t\to\infty$, shows that the integrand behaves like $e^{(1/2-s)t}$. Therefore the condition $\operatorname{Re}(s)>\frac{1}{2}$ is imposed for convergence.

Example 2.5. Entry **7.621.2** states that

$$ \int_0^\infty e^{-st} t^{\mu-1/2} M_{\lambda,\mu}(qt)\, dt = q^{\mu+1/2}\Gamma(2\mu+1)\left(s-\frac{q}{2}\right)^{\lambda-\mu-1/2}\left(s+\frac{q}{2}\right)^{-\lambda-\mu-1/2}, $$

for $\operatorname{Re}(\mu)>-\frac{1}{2}$ and $\operatorname{Re}(s)>\frac{1}{2}|\operatorname{Re}(q)|$.

Proof. Assume first that $q>0$. The change of variables $w=qt$ gives

$$ \int_0^\infty e^{-(s+q/2)t} (qt)^{\mu+1/2} t^{\mu-1/2} {}_1F_1(\mu-\lambda+\tfrac{1}{2}; 2\mu+1; qt)\, dt = \tag{2.20} $$

$$ \frac{1}{q^{\mu+1/2}} \int_0^\infty e^{-(s/q+1/2)w} w^{(2\mu+1)-1} {}_1F_1(\mu-\lambda+\tfrac{1}{2}; 2\mu+1; w)\, dw. $$

The evaluation of this last integral uses Entry **7.621.5** with $c\mapsto 2\mu+1$, $a\mapsto\mu-\lambda+\frac{1}{2}$, $s\mapsto\frac{s}{q}+\frac{1}{2}$. This requires $\operatorname{Re}(c)>1$; that is, $\operatorname{Re}(\mu)>-\frac{1}{2}$ and also $\operatorname{Re}(s)>1$ that translates to $\operatorname{Re}(s)>q/2$. This gives the stated result.

This result is now extended to $q \in \mathbb{C}$ by analytic continuation. The proof uses the asymptotic expansion (1.9) for $z = qt$, where $-\frac{1}{2}\pi < \arg q < \frac{3}{2}\pi$. Note that when $\operatorname{Re} q > 0$, the first term in the asymptotic expansion is dominant and convergence of the resulting integral requires the restriction $\operatorname{Re} s > \operatorname{Re}(q/2)$. Since the right-hand side is also analytic in the region $\operatorname{Re} s > \frac{1}{2}\operatorname{Re} q > 0$, analytic continuation established the formula. Similar argument can be made for $\operatorname{Re} q \leqslant 0$. In the case $\operatorname{Re} q < 0$, the leading term in (1.9) is now the second one. The details are omitted.

Example 2.6. Entry **7.621.3** states that for $\operatorname{Re}\left(\alpha \pm \mu + \frac{3}{2}\right) > 0$, $\operatorname{Re}(s) > -\frac{q}{2}$ and $q > 0$,

(2.21)

$$\begin{aligned}\int_0^\infty e^{-st} t^\alpha W_{\lambda,\mu}(qt)\,dt &= \frac{\Gamma(\alpha+\mu+\frac{3}{2})\Gamma(\alpha-\mu+\frac{3}{2})q^{\mu+\frac{1}{2}}}{\Gamma(\alpha-\lambda+2)}\left(s+\frac{q}{2}\right)^{-\alpha-\mu-\frac{3}{2}} \\ &\times \ {}_2F_1\left(\begin{matrix}\alpha+\mu+\frac{3}{2} \quad \mu-\lambda+\frac{1}{2}\\ \alpha-\lambda+2\end{matrix}\bigg|\frac{2s-q}{2s+q}\right).\end{aligned}$$

Note that from (1.7) and (1.10),

$$W_{\lambda,\mu}(x) = x^{\mu+\frac{1}{2}} e^{-x/2}\Psi(\mu-\lambda+\tfrac{1}{2}; 2\mu+1; x). \tag{2.22}$$

The function $\Psi(a;c;t)$ is defined in (**??**).

The evaluation begins with the change of variables $x = qt$ to produce

$$\begin{aligned}\int_0^\infty e^{-st} t^\alpha W_{\lambda,\mu}(qt)\,dt &= \frac{1}{q^{\alpha+1}}\int_0^\infty e^{-sx/q} x^\alpha W_{\lambda,\mu}(x)\,dx \\ &= \frac{1}{q^{\alpha+1}}\int_0^\infty e^{-\left(\frac{s}{q}+\frac{1}{2}\right)x} x^{(\alpha+\mu+\frac{3}{2})-1}\Psi(\mu-\lambda+\tfrac{1}{2}; 2\mu+1; x)\,dx \\ &= \frac{1}{q^{\alpha+1}}\frac{\Gamma(\alpha+\mu+\frac{3}{2})\Gamma(\alpha-\mu+\frac{3}{2})}{\Gamma(\alpha-\lambda+2)} \\ &\qquad \times {}_2F_1\left(\begin{matrix}\alpha+\mu+\frac{3}{2} \quad \alpha-\mu+\frac{3}{2}\\ \alpha-\lambda+2\end{matrix}\bigg|\frac{1}{2}-\frac{s}{q}\right),\end{aligned}$$

using the first part of **7.621.6** in Example 2.3. The application of formula **7.621.6** requires the conditions

$$\operatorname{Re}(\alpha+\mu+\tfrac{3}{2}) > 0,\ \operatorname{Re}(\alpha-\mu+\tfrac{3}{2}) > 0 \text{ and } \left|\frac{1}{2}-\frac{s}{q}\right| < 1. \tag{2.23}$$

The last condition is more restrictive than the conditions given for the present entry. This can be relaxed to $\operatorname{Re}\left(\frac{s}{q}\right) > -\frac{1}{2}$; that is, $\operatorname{Re}(s) > -\frac{q}{2}$ using (1.11). This shows that the integrand behaves like $e^{-(s/q+1/2)x}$ at infinity. Convergence requires the stated restriction $\operatorname{Re}(s/q+1/2) > 0$. The final form of the answer can now be produced by using Pfaff's transformation.

The special case $\alpha = \nu - 1$, $s = \frac{1}{2}$ and $q = 1$ produces

$$\int_0^\infty e^{-x/2} x^{\nu-1} W_{\kappa,\mu}(x)\,dx = \frac{\Gamma\left(\nu+\frac{1}{2}-\mu\right)\Gamma\left(\nu+\frac{1}{2}+\mu\right)}{\Gamma(\nu-\kappa+1)}. \tag{2.24}$$

This is Entry **7.621.11**. Observe that, given the specialized parameters, the hypergeometric term in **7.621.3** reduces to 1.

Example 2.7. Entry **7.621.7** is evaluated next. This evaluation will show that the answer stated in [**2**] contains a typo. The entry, as stated in the table, is

$$\int_0^\infty e^{-\frac{b}{2}x} x^{\nu-1} M_{\kappa,\mu}(bx)\,dx = \frac{\Gamma(1+2\mu)\Gamma(\kappa-\nu)\Gamma(\frac{1}{2}+\mu+\nu)}{\Gamma(\frac{1}{2}+\mu+\kappa)\Gamma(\frac{1}{2}+\mu-\nu)} b^{\nu}, \tag{2.25}$$

for $\text{Re}\,(\nu+\mu+\frac{1}{2}) > 0$ and $\text{Re}\,(\kappa-\nu) > 0$.

Proof. Assume first that $b > 0$. Then

$$\begin{aligned}
\int_0^\infty e^{-\frac{b}{2}x} x^{\nu-1} M_{\kappa,\mu}(bx)\,dx &= \frac{1}{b^\nu}\int_0^\infty e^{-t/2} t^{\nu-1} M_{\kappa,\mu}(t)\,dt \\
&= \frac{1}{b^\nu}\int_0^\infty e^{-t/2} t^{\nu-1} t^{\mu+\frac{1}{2}} e^{-t/2} {}_1F_1(\mu-\kappa+\tfrac{1}{2}; 2\mu+1; t)\,dt \\
&= \frac{1}{b^\mu}\int_0^\infty e^{-t} t^{(\mu+\nu+\frac{1}{2})-1} {}_1F_1(\mu-\kappa+\tfrac{1}{2}; 2\mu+1; t)\,dt.
\end{aligned}$$

The convergence at $t = 0$ requires $\text{Re}\,(\mu+\nu+\frac{1}{2}) > 0$ and near infinity observe that

$${}_1F_1(\mu-\kappa+\tfrac{1}{2}; 2\mu+1; t) \sim \frac{\Gamma(2\mu+1)}{\Gamma(\mu-\kappa+\frac{1}{2})} e^t t^{-\kappa-\mu-\frac{1}{2}} \sum_{n=0}^\infty \frac{(\mu+\kappa+\frac{1}{2})_n(-\mu+\kappa+\frac{1}{2})_n}{n!} t^{-n}. \tag{2.26}$$

Since the exponential factors cancel, the leading order term at infinity is $t^{\nu-\kappa-1}$, therefore convergence requires $\text{Re}\,(\kappa-\nu) > 0$.

The evaluation of the integral is obtained using the first part of entry **7.621.4** in Example 2.1. This gives

$$\frac{1}{b^\mu}\int_0^\infty e^{-t} t^{(\mu+\nu+\frac{1}{2})-1} {}_1F_1(\mu-\kappa+\tfrac{1}{2}; 2\mu+1; t)\,dt = \frac{\Gamma(\mu+\nu+\frac{1}{2})}{b^\nu} {}_2F_1\left(\left. \begin{matrix} \mu-\kappa+\frac{1}{2} & \mu+\nu+\frac{1}{2} \\ & 2\mu+1 \end{matrix} \right| 1 \right). \tag{2.27}$$

The value of the hypergeometric function is obtained using Gauss formula

$${}_2F_1\left(\left. \begin{matrix} a & b \\ & c \end{matrix} \right| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.28}$$

valid for $\text{Re}\,(c-a-b) > 0$. In the case considered here, this condition becomes $\text{Re}\,(\kappa-\nu) > 0$ which is satisfied. To complete the evaluation, the restriction $b > 0$ is now removed by analytic continuation.

The correct value of the entry is

$$\int_0^\infty e^{-\frac{b}{2}x} x^{\nu-1} M_{\kappa,\mu}(bx)\,dx = \frac{\Gamma(1+2\mu)\Gamma(\kappa-\nu)\Gamma(\frac{1}{2}+\mu+\nu)}{\Gamma(\frac{1}{2}+\mu+\kappa)\Gamma(\frac{1}{2}+\mu-\nu)} b^{-\nu}, \tag{2.29}$$

i.e., the exponent of the parameter b has an error in (2.25).

Example 2.8. Entry **7.621.8** states that, for $\text{Re}\,(\mu+\frac{1}{2}) > 0$ and $\text{Re}\,(s) > \frac{1}{2}$,

$$\int_0^\infty e^{-sx} M_{\kappa,\mu}(x)\,\frac{dx}{x} = \frac{2\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu+\kappa)} e^{-\pi i \kappa} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}} \right)^{\kappa/2} Q^{\kappa}_{\mu-\frac{1}{2}}(2s). \tag{2.30}$$

Proof. Start by using the definition of $M_{\kappa,\mu}(x)$ to obtain

$$\int_0^\infty e^{-sx} M_{\kappa,\mu}(x)\,\frac{dx}{x} = \int_0^\infty e^{-(s+\frac{1}{2})x} x^{\mu-\frac{1}{2}} {}_1F_1(\mu-\kappa+\tfrac{1}{2}; 2\mu+1; x)\,dx. \tag{2.31}$$

The behavior of the integrand at 0 requires $\text{Re}\,(\mu+\frac{1}{2}) > 0$ and at infinity $\text{Re}\,(s) > \frac{1}{2}$.

To produce the evaluation of this entry, assume first $\left| s - \frac{1}{2} \right| > 1$, in order to use the second part of **7.621.4** in Example 2.1, with $s \mapsto s+\frac{1}{2}$, $b \mapsto \mu+\frac{1}{2}$, $k=1$, $a \mapsto \mu-\kappa+\frac{1}{2}$ and $c = 2\mu+1$. This produces

$$\int_0^\infty e^{-sx} M_{\kappa,\mu}(x)\,\frac{dx}{x} = \Gamma(\mu+\tfrac{1}{2})(s-\tfrac{1}{2})^{-\mu-\frac{1}{2}} {}_2F_1 \left(\begin{matrix} \mu+\kappa+\frac{1}{2} & \mu+\frac{1}{2} \\ \multicolumn{2}{c}{2\mu+1} \end{matrix} \middle| \frac{2}{1-2s} \right). \tag{2.32}$$

Now use the transformation [**1**, p. 127, equation (3.1.7)]

$${}_2F_1 \left(\begin{matrix} a & b \\ \multicolumn{2}{c}{2a} \end{matrix} \middle| x \right) = \left(1 - \frac{x}{2}\right)^{-b} {}_2F_1 \left(\begin{matrix} \frac{b}{2} & \frac{b+1}{2} \\ \multicolumn{2}{c}{a+\frac{1}{2}} \end{matrix} \middle| \left(\frac{x}{2-x} \right)^2 \right) \tag{2.33}$$

with $a \mapsto \mu+\frac{1}{2}$, $b \mapsto \mu+\kappa+\frac{1}{2}$ and $x = 2/(1-2s)$ to obtain

$$\begin{aligned} &\int_0^\infty e^{-sx} M_{\kappa,\mu}(x)\,\frac{dx}{x} \\ &\quad = \Gamma(\mu+\tfrac{1}{2})(s-\tfrac{1}{2})^{\kappa} s^{-\mu-\kappa-\frac{1}{2}} {}_2F_1 \left(\begin{matrix} \frac{1}{2}(\mu+\kappa+\frac{1}{2}) & \frac{1}{2}(\mu+\kappa+\frac{3}{2}) \\ \multicolumn{2}{c}{\mu+1} \end{matrix} \middle| \frac{1}{4s^2} \right). \end{aligned} \tag{2.34}$$

The duplication formula for the gamma function

$$\Gamma(2u) = \frac{2^{2u-1}}{\sqrt{\pi}} \Gamma(u)\Gamma(u+\tfrac{1}{2}) \tag{2.35}$$

(entry **8.335.1** in [**2**]) to obtain

$$\begin{aligned} &\int_0^\infty e^{-sx} M_{\kappa,\mu}(x)\,\frac{dx}{x} \\ &\qquad = \frac{2\Gamma(2\mu+1)e^{-i\pi\kappa}}{\Gamma(\mu+\kappa+\frac{1}{2})} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}} \right)^{\kappa/2} \\ &\left\{ \frac{\sqrt{\pi} e^{i\pi\kappa} \Gamma(\mu+\kappa+\frac{1}{2})}{2^{\mu+\frac{1}{2}}\,\Gamma(\mu+1)} (4s^2-1)^{\kappa/2} (2s)^{-(\mu-\frac{1}{2})-\kappa-1} {}_2F_1 \left(\begin{matrix} \frac{1}{2}(\mu+\kappa+\frac{1}{2}) & \frac{1}{2}(\mu+\kappa+\frac{3}{2}) \\ \multicolumn{2}{c}{\mu+1} \end{matrix} \middle| \frac{1}{4s^2} \right) \right\}. \end{aligned} \tag{2.36}$$

The convergence of the hypergeometric term requires $|s| > \frac{1}{2}$. But, as it has been stated before, the convergence of the integral requires a more restrictive condition $\text{Re}\,(s) > \frac{1}{2}$.

The function $Q_{\mu}^{\kappa}(s)$ is called the *associated Legendre function of the second kind* and is defined in entry **8.703** of [**2**, p. 959] as
(2.37)

$$Q_{\nu}^{\mu}(z) = \frac{e^{i\pi\mu}\Gamma(\nu+\mu+1)\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}(z^2-1)^{\mu/2}z^{-\nu-\mu-1}{}_2F_1\left(\begin{matrix}\frac{1}{2}(\mu+\nu+2) & \frac{1}{2}(\mu+\nu+1)\\ & \nu+\frac{3}{2}\end{matrix}\middle|\frac{1}{z^2}\right).$$

Example 2.9. Entry **7.621.9** states that

$$\int_0^\infty e^{-sx}W_{\kappa,\mu}(x)\frac{dx}{x} = \frac{\pi}{\cos\left(\frac{\pi\mu}{2}\right)}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\kappa/2} P_{\mu-\frac{1}{2}}^{\kappa}(2s) \tag{2.38}$$

for $\text{Re}\left(\frac{1}{2}\pm\mu\right) > 0$, $\text{Re}\,(s) > -\frac{1}{2}$. The function $P_{\nu}^{\mu}(z)$ is defined in Entry **8.702** of [**2**] by

$$P_{\nu}^{\mu}(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left(\begin{matrix}-\nu & \nu+1\\ & 1-\mu\end{matrix}\middle|\frac{1-z}{2}\right). \tag{2.39}$$

Proof. Start with the expression

$$W_{\kappa,\mu}(x) = e^{-x/2}x^{\mu+\frac{1}{2}}\Psi(\mu-\kappa+\tfrac{1}{2};2\mu+1;x) \tag{2.40}$$

to obtain

$$\int_0^\infty e^{-sx}W_{\kappa,\mu}(x)\frac{dx}{x} = \int_0^\infty e^{-(s+\frac{1}{2})x}x^{(\mu+\frac{1}{2})-1}\Psi(\mu-\kappa+\tfrac{1}{2};2\mu+1;x)\,dx. \tag{2.41}$$

The convergence of the integral at the origin requires $\text{Re}\,(\mu) > -\frac{1}{2}$ and convergence at infinity requires $\text{Re}\,(s) > -\frac{1}{2}$.

The first formula in Entry **7.621.6** shows that

$$\int_0^\infty e^{-(s+\frac{1}{2})x}x^{(\mu+\frac{1}{2})-1}\Psi(\mu-\kappa+\tfrac{1}{2};2\mu+1;x)\,dx = \frac{\Gamma\left(\frac{1}{2}+\mu\right)\Gamma\left(\frac{1}{2}-\mu\right)}{\Gamma(1-\kappa)}{}_2F_1\left(\begin{matrix}\frac{1}{2}+\mu & \frac{1}{2}-\mu\\ & 1-\kappa\end{matrix}\middle|\frac{1}{2}-s\right). \tag{2.42}$$

The conditions on **7.621.6** require the restriction $\text{Re}\,(\mu) < \frac{1}{2}$.

The expression (2.42) can be written in the form (2.38) by using the elementary identity

$$\Gamma\left(\tfrac{1}{2}+\mu\right)\Gamma\left(\tfrac{1}{2}-\mu\right) = \frac{\pi}{\cos\pi\mu}, \quad \mu-\tfrac{1}{2}\notin\mathbb{Z}, \tag{2.43}$$

that appears as Entry **8.334.2** in [**2**].

Example 2.10. Entry **7.621.10** is

$$\int_0^\infty x^{\kappa+2\mu-1}e^{-3x/2}W_{\kappa,\mu}(x)\,dx = \frac{\Gamma(\kappa+\mu+\frac{1}{2})\,\Gamma\left[\frac{1}{4}(2\kappa+6\mu+5)\right]}{(\kappa+3\mu+\frac{1}{2})\,\Gamma\left[\frac{1}{4}(2\mu-2\kappa+3)\right]} \tag{2.44}$$

under the conditions $\text{Re}\,(k+\mu) > -\frac{1}{2}$, $\text{Re}\,(k+3\mu) > -\frac{1}{2}$.

As usual, the conditions on the parameters can be established by examining the convergence of the integral. The proof begins with

(2.45)
$$\int_0^\infty x^{\kappa+2\mu-1}e^{-3x/2}W_{\kappa,\mu}(x)\,dx = e^{-2x}x^{(3\mu+\kappa+\frac{1}{2})-1}\Psi\left(\mu-\kappa+\tfrac{1}{2};2\mu+1;x\right)\,dx$$

The first part of Entry **7.621.6** gives the value of the integral as

$$\frac{\Gamma(3\mu+\kappa+\frac{1}{2})\,\Gamma(\mu+\kappa+\frac{1}{2})}{\Gamma(2\mu+1)}\,{}_2F_1\left(\begin{matrix}3\mu+\kappa+\frac{1}{2} & \mu+\kappa+\frac{1}{2}\\ & 1+2\mu\end{matrix}\Bigg|-1\right). \tag{2.46}$$

The form of the answer given in [**2**] is obtained by using Kummer's theorem (see Rainville [**3**, p. 68]

$${}_2F_1\left(\begin{matrix}a & b\\ a-b+1 & \end{matrix}\Bigg|-1\right) = \frac{\Gamma(a-b+1)\Gamma(1+\frac{a}{2})}{\Gamma(1+\frac{a}{2}-b)\,\Gamma(1+a)}. \tag{2.47}$$

Example 2.11. The final entry established here is **7.621.12**. It states that, for $\text{Re}\left(\nu+\frac{1}{2}\pm\mu\right)>0$ and $\text{Re}\,(\kappa+\nu)<0$,

$$\int_0^\infty e^{x/2}x^{\nu-1}W_{\kappa,\mu}(x)\,dx = \frac{\Gamma(-\kappa-\mu)\,\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}-\mu+\nu\right)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)\,\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}. \tag{2.48}$$

The evaluation comes directly from Entry **7.612.2**. The convergence at $x=0$ requires $\text{Re}\left(\nu+\mu+\frac{1}{2}\right)>0$ and at infinity $\psi(\mu-\kappa+\frac{1}{2};2\mu+1;x)\sim x^{-\mu+\kappa-1/2}$ and this shows that the integrand is asymptotic to $x^{\nu+\kappa-1}$. Therefore $\text{Re}\,(\kappa+\nu)<0$ is needed for convergence.

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