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Evaluation of entries in Gradshteyn and Ryzhik employing the method of brackets

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Abstract. The method of brackets was created for the evaluation of definite integrals appearing in the resolution of Feynman diagrams. This method consists of a small number of heuristic rules and it is quite easy to apply. The first of these is Ramanujan's Master Theorem, one of his favorite tools to evaluate integrals. The current work illustrates its applicability by evaluating a variety of entries from the classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik.

1. Introduction

The problem of providing a closed-form expression for a definite integral has been studied by a variety of methods. The corresponding problem for indefinite integrals has been solved, for a large class of integrands, by the methods developed by Risch [**20, 21, 22**]. The reader will find in [**7**] a modern description of these ideas and in [**26**] an interesting overwiew of techniques for integration.

The lack of a universal algorithm for the evaluation of definite integrals has created a collection of results saved in the form of Tables of Integrals. The volume created by I. S. Gradshteyn and I. M. Ryzhik [**14**], currently in its *7th* edition, is widely used by the scientific community. Others include [**5, 8, 9, 19**]. The use of symbolic languages, such as `Mathematica` or `Maple`, for this task usually contains a *database search* as a preprocessing of the algorithms. The question of reliability of these tables is essential.

The method of brackets employed here was developed by one of the authors in [**12, 13**] in the context of evaluations of definite integrals obtained from the Schwinger parametrization of Feynman diagrams. The method is closely related to the so-called *negative dimensional integration method* developed by I. G. Halliday and R. M. Ricotta [**15**] and A. T. Suzuki et al. [**23, 24, 25**]. The reader will find a nice collection of examples in [**3, 4**]. The use of this method in the general framework of definite integrals has appeared in [**10, 11**]. In the present work, the flexibility of the method of brackets is illustrated with the evaluation of a selected list of examples from [**14**].

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With just a few rules, the method can easily be automated. Code has been produced in [**17**] using Sage with calls to Mathematica. Testing this implementation against [**14**] has suggested adjustments to the original set of rules in the method. This modified set of rules is presented here.

The main rule of the method of brackets corresponds to one of Ramanujan's favorite method to evaluate integrals of the form $\int_0^\infty dx\, x^{\nu-1} f(x)$. This is the so-called *Ramanujan's Master Theorem.* It states that if $f(x)$ admits a series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} \varphi(n) \frac{(-x)^n}{n!} \tag{1.1}$$

in a neighborhood of $x = 0$, with $f(0) = \varphi(0) \neq 0$, then

$$\int_0^\infty x^{\nu-1} f(x)\, dx = \Gamma(\nu)\varphi(-\nu). \tag{1.2}$$

The integral is the Mellin transform of $f(x)$ and the term $\varphi(-\nu)$ requires an extension of the function φ, initially defined only for $\nu \in \mathbb{N}$. Details on the natural unique extension of φ are given in [**1**]. Observe that, for $\nu > 0$, the condition $\varphi(0) \neq 0$ guarantees the convergence of the integral near $x = 0$. The proof of Ramanujan's Master Theorem and the precise conditions for its application appear in Hardy [**16**]. The reader will find in [**1**] many other examples.

2. The method of brackets

This is a method that evaluates definite integrals over the half line $[0, \infty)$. The application of the method consists of small number of rules, deduced in heuristic form, some of which are placed on solid ground [**1**].

For $a \in \mathbb{R}$, the symbol

$$\langle a \rangle \mapsto \int_0^\infty x^{a-1}\, dx \tag{2.1}$$

is the *bracket* associated to the (divergent) integral on the right. The symbol

$$\phi_n := \frac{(-1)^n}{\Gamma(n+1)} \tag{2.2}$$

is called the *indicator* associated to the index n. The notation $\phi_{i_1 i_2 \cdots i_r}$, or simply $\phi_{12\cdots r}$, denotes the product $\phi_{i_1}\phi_{i_2}\cdots\phi_{i_r}$.

Rules for the production of bracket series

Rule $\mathbf{P_1}$. Power series appearing in the integrand are converted into *bracket series* by the procedure

$$\sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1} \mapsto \sum_{n \geqslant 0} a_n \langle \alpha n + \beta \rangle. \tag{2.3}$$

Rule $\mathbf{P_2}$. For $\alpha \in \mathbb{C}$, the multinomial power $(a_1 + a_2 + \cdots + a_r)^\alpha$ is assigned the r-dimension bracket series

$$\sum_{n_1 \geqslant 0} \sum_{n_2 \geqslant 0} \cdots \sum_{n_r \geqslant 0} \phi_{n_1\, n_2 \cdots n_r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}. \tag{2.4}$$

Rule $\mathbf{P_3}$. Each representation of an integral by a bracket series has associated an *index of the representation* via

$$\text{index } = \text{number of sums } - \text{ number of brackets.} \tag{2.5}$$

It is important to observe that the index is attached to a specific representation of the integral and not just to integral itself. The experience obtained by the authors using this method suggests that, among all representations of an integral as a bracket series, the one with *minimal index* should be chosen.

Rules for the evaluation of a bracket series

Rule $\mathbf{E_1}$. The one-dimensional bracket series is assigned the value

$$\sum_{n \geqslant 0} \phi_n f(n) \langle an + b \rangle \mapsto \frac{1}{|a|} f(n^*) \Gamma(-n^*), \tag{2.6}$$

where n^* is obtained from the vanishing of the bracket; that is, n^* solves $an + b = 0$. This is precisely the Ramanujan's Master Theorem.

The next rule provides a value for multi-dimensional bracket series of index 0, that is, the number of sums is equal to the number of brackets.

Rule $\mathbf{E_2}$. Assuming the matrix $A = (a_{ij})$ is non-singular, then the assignment is

$$\sum_{n_1 \geqslant 0} \cdots \sum_{n_r \geqslant 0} \phi_{n_1 \cdots n_r} f(n_1, \cdots, n_r) \langle a_{11} n_1 + \cdots + a_{1r} n_r + c_1 \rangle \cdots \langle a_{r1} n_1 + \cdots + a_{rr} n_r + c_r \rangle$$

$$\mapsto \frac{1}{|\det(A)|} f(n_1^*, \cdots n_r^*) \Gamma(-n_1^*) \cdots \Gamma(-n_r^*)$$

where $\{n_i^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets. There is no assignment if A is singular.

Rule $\mathbf{E_3}$. The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded. There is no assignment to a bracket series of negative index.

The next sections offer a variety of examples that illustrate these rules.

3. Examples of index 0

This section contains some integrals from [**14**] that lead to bracket series of index 0. The evaluation of these entries by the method of brackets illustrate the rules described in the previous section.

Example 3.1. Entry **3.310** states the elementary result

$$I = \int_0^\infty e^{-x}\, dx = 1. \tag{3.1}$$

The method of brackets begins with the integral representation

$$e^{-x} = \sum_{n_1=0}^{\infty} \frac{(-x)^{n_1}}{n_1!} \tag{3.2}$$

with its corresponding bracket series

$$\sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \langle n_1 + 1 \rangle = \sum_{n_1=0}^{\infty} \phi_{n_1} \langle n_1 + 1 \rangle, \tag{3.3}$$

and the associated function $f(n_1) \equiv 1$. Therefore, this problem produces one sum and a single bracket giving a sum of index 0. The vanishing of the brackets gives $n_1^* = -1$. Rule E_1 gives the integral as

$$I = \Gamma(n_1^*) = \Gamma(1) = 1. \tag{3.4}$$

Example 3.2. The integrand now involves the Bessel function

$$J_\nu(x) := \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\nu+1+n)} \left(\frac{x}{2}\right)^{2n}. \tag{3.5}$$

The bracket series corresponding to the integral

$$I = \int_0^\infty J_\nu(bx)\, dx \tag{3.6}$$

is

$$S = \left(\frac{b}{2}\right)^\nu \sum_{n \geqslant 0} \phi_{n_1} \frac{1}{\Gamma(\nu+1+n_1)} \frac{b^{2n_1}}{4^{n_1}} \langle 2n_1 + \nu + 1 \rangle. \tag{3.7}$$

This bracket series also has index 0: one sum and one bracket. The vanishing of this bracket yields $n_1^* = -\frac{1}{2}(1+\nu)$. Therefore the integral is assigned the value

$$\begin{aligned} I &= \frac{1}{2}\left(\frac{b}{2}\right)^\nu \frac{b^{2n_1^*}}{2^{2n_1^*}\,\Gamma(\nu+1+n_1^*)} \Gamma(-n_1^*) \\ &= \frac{1}{2}\left(\frac{b}{2}\right)^\nu \frac{b^{-1-\nu}}{2^{-1-\nu}\,\Gamma(\frac{\nu+1}{2})} \Gamma(\tfrac{\nu+1}{2}) \\ &= \frac{1}{b}. \end{aligned}$$

This agrees with entry **6.511.1** in [**14**].

Example 3.2. Entry **6.521.11** gives the identity

$$I = \int_0^\infty x^2 K_1(ax)\, dx = \frac{2}{a^3} \tag{3.8}$$

for $a > 0$. The integrand now involves the modified Bessel function of the second kind. Use of the integral representation

$$K_\nu(x) := \frac{2^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{x^\nu \Gamma\left(\frac{1}{2}\right)} \int_0^\infty \frac{\cos(xt)\, dt}{(t^2+1)^{\nu+\frac{1}{2}}} \tag{3.9}$$

produces the double integral

$$I = \int_0^\infty x^2 K_1(ax)\, dx = \int_0^\infty \int_0^\infty \frac{x}{a} \frac{\cos(axt)}{(t^2+1)^{\frac{3}{2}}}\, dt\, dx. \tag{3.10}$$

The $\cos(axt)$ factor is written as a series in n_1

$$\cos(axt) = \sum_{n_1=0}^\infty \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+n_1\right)} \left(-\frac{axt}{2}\right)^{2n_1} \tag{3.11}$$

and Rule P_2 assigns to the factor $(t^2+1)^{-\frac{3}{2}}$ the bracket series

$$\sum_{n_2 \geqslant 0} \sum_{n_3 \geqslant 0} \phi_{n_2\, n_3} t^{2n_2} 1^{n_3} \frac{\langle \frac{3}{2} + n_2 + n_3 \rangle}{\Gamma\left(\frac{3}{2}\right)}. \tag{3.12}$$

The final step in producing the bracket series is to replace $t^{2n_1+2n_2}$ with $\langle 1+2n_1+2n_2 \rangle$ and x^{1+2n_1} with $\langle 2+2n_1 \rangle$. The bracket series

$$\sum_{n_1 \geqslant 0} \sum_{n_2 \geqslant 0} \sum_{n_3 \geqslant 0} \phi_{1,2,3} \frac{2^{1-2n_1} a^{-1+2n_1}}{\Gamma\left(\frac{1}{2}+n_1\right)} \langle 2+2n_1 \rangle \langle 1+2n_1+2n_2 \rangle \langle \tfrac{3}{2} + n_2 + n_3 \rangle \tag{3.13}$$

is of index 0.

The linear system constructed from the vanishing of the brackets is

$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -\frac{3}{2} \end{pmatrix} \tag{3.14}$$

with the matrix A having rank 3 and determinant 4. The solution of the system gives $n_1^* = -1$, $n_2^* = \frac{1}{2}$, and $n_3^* = -2$. The value of the integral by Rule E_2 is

$$\begin{aligned} I &= \frac{1}{4} \left(\frac{2^{1-2n_1^*} a^{-1+2n_1^*}}{\Gamma\left(\frac{1}{2}+n_1^*\right)} \right) \Gamma(-n_1^*)\Gamma(-n_2^*)\Gamma(-n_3^*) \\ &= \frac{1}{4} \left(\frac{2^3 a^{-3}}{\Gamma\left(-\frac{1}{2}\right)} \right) \Gamma(1)\Gamma\left(-\tfrac{1}{2}\right)\Gamma(2) \\ &= \frac{2}{a^3}, \end{aligned}$$

verifying (3.8).

4. Examples of index 1

This section considers integrals that lead to representations of index 1.

Example 4.1 The first example provides an evaluation of the elementary entry **3.311.1** in [**14**]:

$$\int_0^\infty \frac{dx}{e^{px}+1} = \frac{\ln 2}{p}. \tag{4.1}$$

The method of brackets will reduce the problem to a triple series and two brackets leading to a representation of index 1. Rule E_3 reduces the number of sums by two and the answer is expressed as a single series. The remaining series is elementary and is recognized as $\ln 2$.

The first step is to replace the integrand by its brackets series:

$$\sum_{n_1 \geqslant 0} \sum_{n_2 \geqslant 0} \phi_{1,2} (e^{px})^{n_1} 1^{n_2} \frac{\langle 1+n_1+n_2 \rangle}{\Gamma(1)} = \sum_{n_1 \geqslant 0} \sum_{n_2 \geqslant 0} \phi_{1,2} e^{n_1 px} \langle 1+n_1+n_2 \rangle.$$

The power series representation of the exponential is now employed to produce

$$\sum_{n_1 \geqslant 0} \sum_{n_2 \geqslant 0} \phi_{1,2} \left(\sum_{n_3 \geqslant 0} \frac{(xn_1)^{n_3}}{\Gamma(n_3+1)} \right) \langle 1+n_1+n_2 \rangle = \\ \sum_{n_1,n_2,n_3 \geqslant 0} \phi_{1,2,3} (-n_1)^{n_3} p^{n_3} x^{n_3} \langle 1+n_1+n_2 \rangle.$$

This form of integrand produces the bracket series

$$I = \sum_{n_1,n_2,n_3 \geqslant 0} \phi_{1,2,3} (-n_1)^{n_3} p^{n_3} \langle 1+n_1+n_2 \rangle \langle n_3+1 \rangle \tag{4.2}$$

for the integral.

There are now three sums and two brackets, giving a representation of index 1. The matrix equation associated to the vanishing of the brackets

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \tag{4.3}$$

has rank 2. It follows that the problem has 1 free parameter. Observe that the equation coming from the vanishing of the bracket $\langle n_3+1 \rangle$ determines $n_3^* = -1$. The system has reduced to the single equation $n_1+n_2 = -1$. The choices of free indices are n_1 and n_2 and their contributions to the integral are described next.

Case 1: n_1 is free. The relation among the indices yields $n_2^* = -n_1 - 1$ and the corresponding determinant is -1. The contribution of this index to the integral is

$$\sum_{n_1 \geqslant 0} \phi_1 \frac{1}{|-1|} (-n_1 p)^{n_3^*} \Gamma(-n_2^*) \Gamma(-n_3^*) = \sum_{n_1 \geqslant 0} \frac{(-1)^{n_1+1}}{n_1 p}. \tag{4.4}$$

The term $n_1 = 0$ yields the series divergent, so its contribution to the integral is discarded.

Case 2: n_2 is free. Then $n_1^* = -n_2 - 1$ with determinant -1. The contribution of this index to the integral is given by

$$\sum_{n_2 \geqslant 0} \phi_2 \frac{1}{|-1|} (-n_1^* p)^{n_3^*} \Gamma(-n_1^*) \Gamma(-n_3^*) = \sum_{n_2 \geqslant 0} \frac{(-1)^{n_2}}{(n_2+1)p}. \tag{4.5}$$

Adding all the finite contributions of free indices gives the evaluation

$$\int_0^\infty \frac{dx}{e^x+1} = \sum_{n_2 \geqslant 0} \frac{(-1)^{n_2}}{(n_2+1)p}. \tag{4.6}$$

In order to present the integral in its simplest possible form, it is now required to identify this series. In this case this is elementary: the result is

$$\sum_{n_2 \geqslant 0} \frac{(-1)^{n_2}}{(n_2+1)p} = \frac{\ln 2}{p}, \tag{4.7}$$

Thus,

$$\int_0^\infty \frac{dx}{e^{px}+1} = \frac{\ln 2}{p}, \tag{4.8}$$

as stated in [**14**].

Example 4.2. This example illustrates the fact that the method of brackets gives, as the value of a definite integral, a finite number of series. The question of reduction of these series to its simplest form is a separate issue. As of now, there is no algorithmic solution to this question.

Entry **3.452.1** states that

$$I = \int_0^\infty \frac{x\,dx}{\sqrt{e^x-1}} = 2\pi \ln 2. \tag{4.9}$$

The brackets series for the integral is obtained as before, with the result

$$I = \sum_{n_1, n_2 . n_3} \phi_{1,2,3} \frac{1}{\sqrt{\pi}} (-1)^{n_2+n_3} \langle n_3 + 2 \rangle \langle n_1 + n_2 + \tfrac{1}{2} \rangle. \tag{4.10}$$

This is a representation of index 1 (three sums and two brackets).

The vanishing of the brackets shows that n_3 is fixed: $n_3^* = -2$ and the relation $n_1 + n_2 + \frac{1}{2} = 0$ must hold. Therefore the integral is given in terms of a single series.

Case 1: if n_1 is free, then $n_2^* = -n_1 - \frac{1}{2}$ and the corresponding series is

$$S_1 = \sum_{n_1=0}^\infty (-1)^{2n_1+5/2} \frac{\Gamma(n_1+1/2)}{\sqrt{\pi} n_1^2 \, \Gamma(n_1+1)}. \tag{4.11}$$

This series is discarded due to the presence of the singular term at $n_1 = 0$.

Case 2: if n_2 is free, then $n_1^* = -n_2 - \frac{1}{2}$ and the corresponding series is

$$S_2 = \sum_{n_2=0}^\infty \frac{4\Gamma(n_2+1/2)}{(2n_2+1)^2 \sqrt{\pi} \, \Gamma(n_2+1)}. \tag{4.12}$$

The duplication formula for the gamma function

$$\Gamma(m+\tfrac{1}{2}) = \frac{(2m)!}{2^{2m}\,m!}\sqrt{\pi} \tag{4.13}$$

reduces the series to

$$S_2 = \sum_{n_2=0}^{\infty} \binom{2n_2}{n_2} \frac{2^{-2n_2}}{(2n_2+1)^2}. \tag{4.14}$$

The method of brackets now yields

$$\int_0^\infty \frac{x\,dx}{\sqrt{e^x-1}} = \sum_{n_2=0}^{\infty} \binom{2n_2}{n_2} \frac{2^{-2n_2}}{(2n_2+1)^2}. \tag{4.15}$$

To evaluate the series start with

$$\sum_{m=0}^{\infty} \binom{2m}{m} x^m = \frac{1}{\sqrt{1-4x}}, \tag{4.16}$$

replace x by x^2 and integrate from 0 to 1/2 to produce (after simplifications)

$$\sum_{m=0}^{\infty} \frac{\binom{2m}{m}}{(2m+1)^2} 2^{-2m} = \int_0^1 \frac{\operatorname{Arcsin} u}{u}\,du. \tag{4.17}$$

Finally, integrate by parts to obtain

$$\sum_{m=0}^{\infty} \frac{\binom{2m}{m}}{(2m+1)^2} 2^{-2m} = -\int_0^{\pi/2} \ln \sin y\,dy. \tag{4.18}$$

Euler showed that this last integral evaluates to $-\frac{\pi}{2}\ln 2$. Details of this elementary evaluation can be found in Section 12.5 of [**6**]. Formula (4.9) has been verified.

Example 4.3. Entry **6.554.1** gives the evaluation

$$\int_0^\infty x J_0(xy)\,\frac{dx}{(x^2+a^2)^{3/2}} = a^{-1}e^{-ay} \tag{4.19}$$

for $y>0$ and $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$. Here J_0 is the Bessel function

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k}\,k!^2}. \tag{4.20}$$

The bracket representations of the terms in the integrand are

$$(a^2+x^2)^{-3/2} = \sum_{n_1 \geqslant 0} \sum_{n_2 \geqslant 0} \phi_{12} a^{2n_1} x^{2n_2} \frac{\langle \frac{3}{2}+n_1+n_2 \rangle}{\Gamma\left(\frac{3}{2}\right)} \tag{4.21}$$

and

$$J_0(xy) = \sum_{n_3 \geqslant 0} \phi_{n_3} \frac{1}{2^{2n_3}\,\Gamma(1+n_3)} (xy)^{2n_3}. \tag{4.22}$$

Therefore the integral is assigned the bracket series

$$\int_0^\infty xJ_0(xy)\,\frac{dx}{(x^2+a^2)^{3/2}} \mapsto \sum_{n_1\geqslant 0}\sum_{n_2\geqslant 0}\sum_{n_3\geqslant 0}\phi_{1,2,3}\frac{y^{2n_3}a^{2n_1}}{2^{2n_3}\Gamma(1+n_3)\Gamma(\frac{3}{2})}\langle \tfrac{3}{2}+n_1+n_2\rangle\,\langle 2n_2+2n_3+2\rangle. \tag{4.23}$$

This is a representation of index $+1$.

Case 1: n_1 free. The linear system from the vanishing of brackets is

$$\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} -n_1-\frac{3}{2} \\ -2 \end{pmatrix} \tag{4.24}$$

with determinant 2 and solutions $n_2^* = -n_1-\frac{3}{2}$ and $n_3^* = n_1+\frac{1}{2}$. The contribution to the integral is given by

$$\begin{aligned} S_1 &= \sum_{n_1\geqslant 0}\frac{(-1)^{n_1}2^{-2n_1-1}a^{2n_1}y^{2n_1+1}\Gamma(-n_1-\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n_1+1)} \\ &= -y\sum_{n_1=0}^{\infty}\frac{(ay)^{2n_1}}{(2n_1+1)!} \\ &= -\frac{\sinh ay}{a}. \end{aligned}$$

Case 2: n_2 free. Proceeding as before it is found that this case leads to a divergent series so its contribution is ignored.

Case 3: n_3 free. As in Case 1, the system has determinant -2 with solutions $n_1^* = n_3-\frac{1}{2}$ and $n_2^* = -n_3-\frac{1}{2}$. The contribution to the integral is

$$\begin{aligned} S_3 &= \sum_{n_3\geqslant 0}\frac{(-1)^{n_3}a^{2n_3-1}y^{2n_3}\Gamma(-n_3+\frac{1}{2})}{\sqrt{\pi}\,2^{2n_3}\,\Gamma(n_3+1)} \\ &= \frac{\cosh ay}{a}. \end{aligned}$$

Summing the finite contributions by Rule E_3 gives

$$\int_0^\infty xJ_0(xy)\,\frac{dx}{(x^2+a^2)^{3/2}} = S_1+S_3 = \frac{e^{-ay}}{a}, \tag{4.25}$$

as stated.

Example 4.4. Entry **6.512.1** provides the value for the integral

$$\int_0^\infty J_\mu(ax)J_\nu(bx)\,dx. \tag{4.26}$$

The answer in [**14**] is divided according to conditions on the parameters.

The integral is provided a brackets series by the usual method:

$$\begin{aligned}\int_0^\infty J_\mu(ax)J_\nu(bx)\,dx &= \int_0^\infty \left(\frac{\left(\frac{ax}{2}\right)^\mu}{\Gamma(\mu+1)} \sum_{n_1=0}^\infty \frac{\left(-\frac{(ax)^2}{4}\right)^{n_1}}{(\mu+1)_{n_1}\, n_1!} \right) \times \\ & \qquad \left(\frac{\left(\frac{bx}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{n_2=0}^\infty \frac{\left(-\frac{(bx)^2}{4}\right)^{n_2}}{(\nu+1)_{n_2}\, n_2!} \right) dx\end{aligned}$$

is given by the series

$$S = \frac{a^\mu b^\nu}{2^{\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n_1,n_2} \phi_{12} \frac{a^{2n_1}\, b^{2n_2}}{2^{2n_1+2n_2}\,(\mu+1)_{n_1}(\nu+1)_{n_2}} \langle 2n_1+2n_2+\mu+\nu+1\rangle.$$

This is a representation of index 1.

Case 1: n_2 free. Then $n_1^* = -\frac{1}{2}(2n_2+\mu+\nu+1)$ and the contribution to the integral is

$$\begin{aligned} S_1 &= \frac{a^\mu b^\nu}{2^{\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n_2\geqslant 0} \phi_2 \frac{b^{2n_2}}{(\nu+1)_{n_2} 2^{2n_2}} \left(\frac{1}{2}\left(\frac{a}{2}\right)^{2n_1^*} \frac{\Gamma(-n_1^*)}{(\mu+1)_{n_1^*}} \right) \\ &= \frac{b^\nu a^{-\nu-1}}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n_2\geqslant 0} \frac{(-1)^{n_2}}{n_2!} \left(\frac{b^2}{a^2}\right)^{n_2} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}+n_2\right)}{(\nu+1)_{n_2}} \frac{\Gamma(\mu+1)}{\Gamma\left(\frac{\mu+1-\nu}{2}-n_2\right)} \\ &= \frac{b^\nu a^{-\nu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)} \sum_{n_2\geqslant 0} \frac{\left(-\frac{b^2}{a^2}\right)^{n_2}}{n_2!} \frac{\left(\frac{\mu+\nu+1}{2}\right)_{n_2}}{(\nu+1)_{n_2}} \frac{1}{\left(\frac{\mu-\nu+1}{2}\right)_{-n_2}\Gamma\left(\frac{\mu+1-\nu}{2}\right)}. \end{aligned}$$

Simplifying the Pochhammer with negative index using

$$(a)_{-n} = \frac{(-1)^n}{(1-a)_n} \tag{4.27}$$

gives

$$\begin{aligned} S_1 &= \frac{b^\nu a^{-\nu-1}\,\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+1-\nu}{2}\right)} \sum_{n_2\geqslant 0} \frac{\left(-\frac{b^2}{a^2}\right)^{n_2}\left(\frac{\mu+\nu+1}{2}\right)_{n_2}\left(1-\frac{\mu-\nu+1}{2}\right)_{n_2}}{n_2!(\nu+1)_{n_2}(-1)^{n_2}} \\ &= \frac{b^\nu a^{-\nu-1}\,\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+1-\nu}{2}\right)} \sum_{n_2\geqslant 0} \frac{\left(\frac{b^2}{a^2}\right)^{n_2}\left(\frac{\mu+\nu+1}{2}\right)_{n_2}\left(\frac{-\mu+\nu+1}{2}\right)_{n_2}}{n_2!(\nu+1)_{n_2}} \\ &= \frac{b^\nu a^{-\nu-1}\,\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+1-\nu}{2}\right)} {}_2F_1\left(\left. \begin{matrix} \frac{\mu+\nu+1}{2},\ \frac{\nu-\mu+1}{2} \\ \nu+1 \end{matrix} \right| \frac{b^2}{a^2} \right). \end{aligned}$$

The series converges provided $|b|<|a|$.

Case 2: n_2 free. The calculation is done as in Case 1. The result is the formula in Case 1, with μ and ν interchanged and a and b interchanged.

Example 4.5. Entry **3.423.1** is

$$\int_0^\infty \frac{x^{\nu-1}\,dx}{(e^x-1)^2} = \Gamma(\nu)\left[\zeta(\nu-1)-\zeta(\nu)\right]. \tag{4.28}$$

The method of brackets provides a direct evaluation. The bracket series corresponding to the integral is

$$\int_0^\infty \frac{x^{\nu-1}\,dx}{(e^x-1)^2} \mapsto \sum_{n_1\geqslant 0}\sum_{n_2\geqslant 0}\sum_{n_3\geqslant 0} \phi_{1,2,3}(-1)^{n_2+n_3} n_1^{n_3}\langle n_3+\nu\rangle\,\langle n_1+n_2+2\rangle.$$

This is a representation of index 1 and $n_3^* = -\nu$ is determined. Choosing n_1 as free parameter gives the series

$$\sum_{n_1\geqslant 0} \frac{(-1)^{\nu-2}\Gamma(n_1+2)\,\Gamma(\nu)}{n_1^\nu\,\Gamma(n_1+1)}.$$

The term $n_1=0$ makes the series diverge, so its contribution is ignored.

The choice of n_2 as a free parameter gives the series

$$\sum_{n_2\geqslant 0} \frac{(-1)^{\nu}\Gamma(n_2+2)\,\Gamma(\nu)}{(-n_2-2)^\nu\,\Gamma(n_2+1)} = \sum_{n_2\geqslant 0}\frac{(n_2+1)\,\Gamma(\nu)}{(n_2+2)^\nu}. \tag{4.29}$$

The answer is obtained by writing $n_2+1=(n_2+2)-1$.

5. Examples of index 2

This section considers integrals that lead to representations of index 2.

Example 5.1. Entry **7.414.9** provides the value

$$\int_0^\infty e^{-x}x^{a+b}L_m^a(x)L_n^b(x)\,dx = (-1)^{m+n}(a+b)!\binom{a+m}{n}\binom{b+n}{m}, \tag{5.1}$$

with $\operatorname{Re} a+b>-1$. Here $L_n^\lambda(x)$ is the associated Laguerre polynomial:

$$L_n^\lambda(z) = \frac{(\lambda+1)_n}{n!}\,{}_1F_1\left(\left.\begin{matrix}-n\\ \lambda+1\end{matrix}\right| z\right) \tag{5.2}$$

The resulting bracket series is of index $+2$:

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\phi_{1,2,3}\frac{(-1)^{n_2+n_3}\Gamma(-n+n_3)\Gamma(-m+n_2)\Gamma(b+n+1)\Gamma(a+m+1)}{\Gamma(n+1)\Gamma(m+1)\Gamma(b+n_3+1)\Gamma(a+n_2+1)\Gamma(-m)\Gamma(-n)} \times\langle a+b+n_1+n_2+n_3+1\rangle.$$

There are three choices of free/fixed variables in the solution of the linear system coming from the vanishing of the bracket:

Case 1: With n_1 and n_2 free, the resulting series is zero and makes no contribution:

$$\begin{aligned}\sum_{n_1,n_2} & \frac{(-1)^{a+b+2n_1+n_2+1}\Gamma(-m+n_2)\Gamma(b+n+1)\Gamma(a+m+1)}{\Gamma(n_2+1)\Gamma(n_1+1)\Gamma(n+1)\Gamma(m+1)\Gamma(a+n_2+1} \\ & \times \frac{\Gamma(a+b+n_1+n_2+1)\Gamma(-a-b-n-n_1-n_2-1)}{)\Gamma(-a-n_1-n_2)\Gamma(-m)\Gamma(-n)} \\ &= \frac{(-1)^{a+b}\Gamma(1+a+b)\Gamma(-m)\Gamma(1+a+m)\Gamma(1+b+n)\Gamma(-a-m+n)}{\pi^2\sin((a+b+n)\pi)\Gamma(1+n)\Gamma(1+b-m+n)} \\ & \qquad \times \sin(a\pi)\sin(m\pi)\sin(n\pi) = 0.\end{aligned}$$

Case 2: With n_1 and n_3 free, the result is the same as in Case 1.

Case 3: With n_2 and n_3 free, the resulting series can be evaluated as follows to match the value in the table:

$$\begin{aligned}&\sum_{n_2}\sum_{n_3} \frac{\Gamma(-n+n_3)\Gamma(-m+n_2)\Gamma(b+n+1)\Gamma(a+m+1)\Gamma(a+b+n_2+n_3+1)}{\Gamma(n_3+1)\Gamma(n_2+1)\Gamma(n+1)\Gamma(m+1)\Gamma(b+n_3+1)\Gamma(a+n_2+1)\Gamma(-m)\Gamma(-n)} \\ &=\sum_{n_2} \frac{\Gamma(-m+n_2)\Gamma(b+n+1)\Gamma(a+m+1)}{\Gamma(n_2+1)\Gamma(n+1)\Gamma(m+1)\Gamma(a+n_2+1)\Gamma(-m)\Gamma(-n)} \\ &\times \frac{\Gamma(-n)\Gamma(a+b+n_2+1)}{\Gamma(b+1)}\,{}_2F_1(-n,1+a+b+n_2;1+b;1) \\ &=\sum_{n_2} \frac{\Gamma(-m+n_2)\Gamma(b+n+1)\Gamma(a+m+1)\Gamma(a+b+n_2+1)}{\Gamma(n_2+1)\Gamma(n+1)\Gamma(m+1)\Gamma(a+n_2+1)\Gamma(-m)\Gamma(b+1)} \frac{\Gamma(1+b)\Gamma(-a+n-n_2)}{\Gamma(1+b+n)\Gamma(-a-n_2)} \\ &=\frac{\Gamma(a+m+1)\Gamma(1+a+b)}{\Gamma(n+1)\Gamma(m+1)} \sum_{n_2} \frac{(-m)_{n_2}(1+a+b)_{n_2}(-\pi\csc\pi(a-n+n_2))}{n_2!(-\pi\csc\pi(a+n_2))\Gamma(1+a-n+n_2)} \\ &=\frac{\Gamma(a+m+1)\Gamma(1+a+b)}{\Gamma(n+1)\Gamma(m+1)\Gamma(1+a-n)} \sum_{n_2} \frac{(-m)_{n_2}(1+a+b)_{n_2}(\sin a\pi)(-1)^{n_2})}{n_2!(\sin a\pi)(-1)^{-n+n_2}(1+a-n)_{n_2}} \\ &=\frac{(-1)^n\Gamma(a+m+1)\Gamma(1+a+b)}{\Gamma(n+1)\Gamma(m+1)\Gamma(1+a-n)}\,{}_2F_1(-m,1+a+b;1+a-n;1) \\ &=\frac{(-1)^n\Gamma(a+m+1)\Gamma(1+a+b)}{\Gamma(n+1)\Gamma(m+1)\Gamma(1+a-n)} \frac{\Gamma(1+a-n)\Gamma(m-n-b)}{\Gamma(1+a-n+m)\Gamma(-n-b)} \\ &=(-1)^{m+n}\Gamma(a+b+1)\frac{\Gamma(a+m+1)}{\Gamma(n+1)\Gamma(a+m-n+1)} \frac{\Gamma(b+n+1)}{\Gamma(m+1)\Gamma(b+n-m+1)}.\end{aligned}$$

6. The goal is to minimize the index

In this section the last part of Rule P_3 is illustrated. Given a specific definite integral, it has been conjectured by the authors that the optimal solution by the method of brackets is the one with minimal index.

Observe that the index of an integral may be affected by the representation of the integrand or the order of expansion into series.

Example 6.1. Entry **3.331.1** gives the evaluation of

$$\int_0^\infty e^{-\beta e^{-x}-\mu x}\,dx = \beta^{-\mu}\gamma(\mu,\beta). \tag{6.1}$$

Here $\gamma(\mu,\beta)$ is the *incomplete gamma function* defined by

$$\gamma(\alpha,x) = \int_0^x e^{-t}t^{\alpha-1}\,dt. \tag{6.2}$$

Method 1. The integrand is associated a bracket series via

$$\begin{aligned} e^{-\beta e^{-x}}e^{-\mu x} &= \sum_{n_1\geqslant 0}\phi_{n_1}(\beta e^{-x})^{n_1}e^{-\mu x} \\ &= \sum_{n_1\geqslant 0}\phi_{n_1}\beta^{n_1}e^{-(n_1+\mu)x} \\ &= \sum_{n_1\geqslant 0}\beta^{n_1}\sum_{n_2\geqslant 0}\phi_{n_2}(n_1+\mu)^{n_2}x^{n_2}. \end{aligned}$$

The final step is to produce the bracket $\langle n_2+1\rangle$ appearing from integration of x^{n_2}. Therefore, the bracket series associated with this representation of the integral is

$$\int_0^\infty e^{-\beta e^{-x}-\mu x}\,dx \mapsto \sum_{n_1\geqslant 0}\sum_{n_2\geqslant 0}\phi_{n_1n_2}\beta^{n_1}(n_1+\mu)^{n_2}\langle n_2+1\rangle. \tag{6.3}$$

This representation has index $+1$.

The vanishing of the bracket yields that n_2 is fixed as $n_2^* = -1$ and n_1 must be free. It follows that the integral is

$$\begin{aligned} \sum_{n_1\geqslant 0}\phi_{n_1}\beta^{n_1}(n_1+\mu)^{-1}\Gamma(1) &= \frac{\Gamma(\mu)}{\Gamma(\mu+1)}\,{}_1F_1\left(\begin{matrix}\mu\\ \mu+1\end{matrix}\middle|\beta\right) \\ &= \frac{1}{\beta^\mu}\gamma(\mu,\beta). \end{aligned}$$

Method 2. A second representation is produced as follows:

$$\begin{aligned} e^{-\beta e^{-x}}e^{-\mu x} &= \sum_{n_1\geqslant 0}\phi_{n_1}(\beta e^{-x})^{n_1}e^{-\mu x} \\ &= \sum_{n_1\geqslant 0}\phi_{n_1}\beta^{n_1}e^{-n_1x}e^{-\mu x} \\ &= \sum_{n_1\geqslant 0}\phi_{n_1}\beta^{n_1}\sum_{n_2\geqslant 0}\phi_{n_2}n_1^{n_2}x^{n_2}\sum_{n_3\geqslant 0}\phi_{n_3}\mu^{n_3}x^{n_3} \\ &= \sum_{n_1\geqslant 0}\sum_{n_2\geqslant 0}\sum_{n_3\geqslant 0}\phi_{n_1,n_2,n_3}\beta^{n_1}n_1^{n_2}\mu^{n_3}x^{n_2+n_3}. \end{aligned}$$

The final step is now to replace the power $x^{n_2+n_3}$ by the bracket $\langle n_2+n_3+1\rangle$ to produce

$$\int_0^\infty e^{-\beta e^{-x}-\mu x}\,dx \mapsto \sum_{n_1\geqslant 0}\sum_{n_2\geqslant 0}\sum_{n_3\geqslant 0}\phi_{n_1,n_2,n_3}\beta^{n_1}n_1^{n_2}\mu^{n_3}\langle n_2+n_3+1\rangle. \tag{6.4}$$

This representation has index $+2$.

The vanishing of the brackets shows that n_1 is free and either n_2 or n_3 is fixed.

Case 1: n_1 free and $n_3^* = -n_2-1$:

$$\begin{aligned}
S_1 &= \sum_{n_1\geqslant 0}\sum_{n_2\geqslant 0}\phi_{n_1,n_2}\beta^{n_1}n_1^{n_2}\mu^{-1-n_2}\Gamma(n_2+1)\\
&= \sum_{n_1\geqslant 0}\frac{(-\beta)^{n_1}}{\mu\Gamma(n_1+1)}\sum_{n_2\geqslant 0}\left(-\frac{n_1}{\mu}\right)^{n_2}\\
&= \sum_{n_1\geqslant 0}\frac{(-\beta)^{n_1}}{\mu\Gamma(n_1+1)}\left(1+\frac{n_1}{\mu}\right)^{-1}\\
&= \frac{\Gamma(\mu)}{\Gamma(\mu+1)}\,{}_1F_1\left(\left.\begin{matrix}\mu\\ \mu+1\end{matrix}\right|-\beta\right)\\
&= \beta^{-\mu}\gamma(\mu,\beta).
\end{aligned}$$

Case 2: n_1 free and $n_2^* = -n_3-1$:

$$\begin{aligned}
S_2 &= \sum_{n_1\geqslant 0}\sum_{n_3\geqslant 0}\phi_{n_1,n_3}\beta^{n_1}n_1^{-n_3-1}\mu^{n_3}\Gamma(n_3+1)\\
&= \sum_{n_1\geqslant 0}\frac{(-\beta)^{n_1}}{n_1\Gamma(n_1+1)}\sum_{n_3\geqslant 0}\left(-\frac{\mu}{n_1}\right)^{n_3}\\
&= \sum_{n_1\geqslant 0}\frac{(-\beta)^{n_1}}{n_1\Gamma(n_1+1)}\left(1+\frac{\mu}{n_1}\right)^{-1}\\
&= \frac{\Gamma(\mu)}{\Gamma(\mu+1)}\,{}_1F_1\left(\left.\begin{matrix}\mu\\ \mu+1\end{matrix}\right|-\beta\right)\\
&= \beta^{-\mu}\gamma(\mu,\beta).
\end{aligned}$$

Rule E_3 would return the sum S_1+S_2, but this is *twice* the correct value. Among these two methods for producing bracket series, the one giving the minimal index should be chosen. This doubling phenomena has appearing in other examples where the minimal index is not chosen. The reason behind this phenomena remains to be elucidated.

Example 6.2. Entry **3.451.1** in [**14**] states that

$$\int_0^\infty xe^{-x}\sqrt{1-e^{-x}}\,dx = \frac{4}{9}\left(4-3\ln 2\right). \tag{6.5}$$

To evaluate this entry by classical methods, observe that the integral is $-h'(1)$, where

$$h(a) = \int_0^\infty e^{-ax}\sqrt{1-e^{-x}}\,dx. \tag{6.6}$$

The change of variables $t = e^{-x}$ gives

$$h(a) = \int_0^1 t^{a-1}(1-t)^{1/2}\,dt = B(a, \tfrac{3}{2}), \tag{6.7}$$

where B is the beta function. Differentiation yields

$$h'(a) = h(a)\left[\psi(a) - \psi(a+\tfrac{3}{2})\right], \tag{6.8}$$

where $\psi = \Gamma'/\Gamma$ is the polygamma function. This gives

$$\int_0^\infty xe^{-x}\sqrt{1-e^{-x}}\,dx = -\frac{\Gamma(1)\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})}\left[\psi(1)-\psi(\tfrac{5}{2})\right]. \tag{6.9}$$

The values $\psi(1) = -\gamma$ (the Euler constant) and $\psi(\frac{5}{2}) = -\gamma - 2\ln 2 + \frac{8}{3}$ give the result. To obtain this last special value use

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k} \text{ and } 2\psi(2x) = 2\ln 2 + \psi(x) + \psi(x+\tfrac{1}{2}). \tag{6.10}$$

This last relation follows by differentiaton of the duplication formula for the gamma function $\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})/\sqrt{\pi}$.

The evaluation of (6.5) is now obtained by the method of brackets.

Method 1. The exponential term is replaced by

$$e^{-x} = \sum_{n=0}^\infty \frac{(-1)^n x^n}{n!} \mapsto \sum_{n_1 \geqslant 0}\phi_1 x^{n_1} \tag{6.11}$$

and Rule P_2 is employed to produce

$$\sqrt{1-e^{-x}} = \sum_{n_2\geqslant 0}\sum_{n_3 \geqslant 0}\phi_{2,3}1^{n_2}(-e^{-x})^{n_3}\frac{\langle -\frac{1}{2}+n_2+n_3\rangle}{\Gamma(-1/2)}. \tag{6.12}$$

Now expand the exponential terms e^{-n_3x} and replace the integral by the corresponding bracket to obtain the series

$$\sum_{n_1\geqslant 0}\sum_{n_2\geqslant 0}\sum_{n_3\geqslant 0}\sum_{n_4\geqslant 0}\phi_{1,2,3,4}\frac{(-1)^{n_3}n_3^{n_4}}{\Gamma(-1/2)}\langle n_1+n_4+2\rangle\langle n_2+n_3-\tfrac{1}{2}\rangle. \tag{6.13}$$

This gives a representation of index $+2$.

Case 1: n_1, n_2 free. Then $n_4^* = -n_1 - 2$ and $n_3^* = -n_2 + \frac{1}{2}$. The corresponding determinant is -1 and the series becomes

$$\sum_{n_1\geqslant 0}\sum_{n_2\geqslant 0}\frac{-(-n_2+1/2)^{-n_1-2}(-1)^{n_1+1/2}\Gamma(n_2-1/2)\Gamma(n_1+2)}{2\sqrt{\pi}\Gamma(n_2+1)\Gamma(n_1+1)}$$

This result is *purely imaginary* and therefore discarded.

Case 2: n_1, n_3 free. Then $n_4^* = -n_1 - 2$ and $n_2^* = -n_3 + \frac{1}{2}$. The determinant is -1 and the series becomes

$$\begin{aligned}
S_2 &= \sum_{n_1} \sum_{n_3} \frac{-(-1)^{n_1} n_3^{-n_1-2} \Gamma(n_3 - 1/2)\Gamma(n_1+2)}{2\sqrt{\pi}\Gamma(n_3+1)\Gamma(n_1+1)} \\
&= \sum_{n_3} \frac{-\Gamma(n_3 - 1/2)}{2\sqrt{\pi}\Gamma(n_3+1)} \sum_{n_1} \frac{(-n_3)^{n_1}\Gamma(n_1+2)}{\Gamma(n_1+1)} \\
&= \sum_{n_3} \frac{-\Gamma(n_3-1/2)}{2\sqrt{\pi}\Gamma(n_3+1)(1+n_3)^2} \\
&= {}_3F_2\left(\left. \begin{matrix} 1, 1, -\frac{1}{2} \\ 2, 2 \end{matrix} \right| 1 \right) \\
&= \frac{\psi\left(\frac{5}{2}\right) + \gamma}{\frac{3}{2}} \\
&= \tfrac{4}{9}\left(4 - 3\ln 2\right).
\end{aligned}$$

Case 3: n_2, n_4 free. This case is similar to Case 1 so it is also discared.
Case 4: n_3, n_4 free. This case is analog to Case 2 with value $\frac{2}{3}\left(\frac{8}{3} - \ln 4\right)$.

Summing the results from Cases 2 and 4 would result in *twice* the correct value. This doubling is due to the fact that the series in each of these cases converge on the boundary.

Method 2. An alternative is to obtain the bracket series for $(1-e^{-x})^{-1/2}$ to produce

$$\int_0^\infty xe^{-x}(1-e^{-x})^{-1/2}\,dx \mapsto \int_0^\infty xe^{-x} \sum_{n_1 \geqslant 0} \sum_{n_2 \geqslant 0} \phi_{1,2} 1^{n_1} (-e^{-x})^{n_2} \frac{\langle -\frac{1}{2} + n_1 + n_2 \rangle}{\Gamma(-1/2)}\,dx,$$

that can be written as

$$\int_0^\infty xe^{-x}(1-e^{-x})^{-1/2}\,dx \mapsto \int_0^\infty \sum_{n_1 \geqslant 0} \sum_{n_2 \geqslant 0} \phi_{1,2} (-1)^{n_2} x e^{-(1+n_2)x} \frac{\langle -\frac{1}{2} + n_1 + n_2 \rangle}{\Gamma(-1/2)}\,dx.$$

The exponential term is now expanded to produce the representation

$$\int_0^\infty xe^{-x}(1-e^{-x})^{-1/2}\,dx \mapsto \frac{1}{\Gamma(-1/2)} \sum_{n_1 \geqslant 0} \sum_{n_2 \geqslant 0} \sum_{n_2 \geqslant 0} \phi_{1,2,3} (-1)^{n_2} (1+n_2)^{n_3} \langle n_3 + 2 \rangle \langle n_1 + n_2 - \tfrac{1}{2} \rangle.$$

This is a representation of index $+1$.

The value $n_3^* = -2$ is determined and the indices n_1 and n_2 are free.

Case 1. n_1 is free. Then $n_2^* = \frac{1}{2} - n_1$ leads to the contribution

$$\sum_{n_1 \geqslant 0} \phi_1 \frac{(-1)^{\frac{1}{2} - n_1} (\frac{3}{2} - n_1)^{-2}}{\Gamma(-1/2)} \Gamma(2)\Gamma(n_1 - \tfrac{1}{2}). \tag{6.14}$$

This is discarded because it is purely imaginary.

Case 2. n_2 is free. Then $n_1^* = \frac{1}{2} - n_2$ produces to the contribution

$$\sum_{n_2 \geqslant 0} \phi_2 \frac{(-1)^{n_2}(1+n_2)^{-2}}{\Gamma(-1/2)} \Gamma(2)\Gamma(n_2 - \tfrac{1}{2}). \tag{6.15}$$

Therefore, the method of brackets shows that

$$\begin{aligned}
\int_0^\infty xe^{-x}(1-e^{-x})^{-1/2}\, dx &= -\frac{1}{2\sqrt{\pi}} \sum_{n_2 \geqslant 0} \phi_2 \frac{(-1)^{n_2}\Gamma(n_2 - \frac{1}{2})}{(1+n_2)^2} \\
&= {}_3F_2\left(\left. \begin{matrix} 1, 1, \frac{1}{2} \\ 2, 2 \end{matrix} \right| 1 \right) \\
&= \frac{\psi\left(\frac{5}{2}\right) + \gamma}{\frac{3}{2}} \\
&= \tfrac{4}{9}\left(4 - 3\ln 2\right).
\end{aligned}$$

This verifies (6.5). As in the first method for this integral, the series converged on the boundary, but it was counted only once in this evaluation using the bracket series of index +1.

7. The evaluation of a Mellin transform

Several entries in [**14**] are instances of the Mellin transform

$$\mathcal{M}(f) := \int_0^\infty x^{s-1} f(x)\, dx. \tag{7.1}$$

For instance, entry **3.764.2**:

$$\int_0^\infty x^p \cos(ax+b)\, dx = -\frac{1}{a^{p+1}} \Gamma(p+1) \sin\left(b + \frac{\pi p}{2}\right), \tag{7.2}$$

is of this form with $p = s - 1$. The reader will find in [**2**] an elementary proof of this evaluation.

The evaluation of (7.2) by the method of brackets uses the hypergeometric representation

$$\cos(ax+b) = {}_0F_1\left(\left. \begin{matrix} - \\ \frac{1}{2} \end{matrix} \right| -\frac{(ax+b)^2}{4} \right) = \sum_{n_1=0}^\infty \frac{\Gamma(\frac{1}{2})}{n_1!\,\Gamma(n+\frac{1}{2})} \left(-\frac{(ax+b)^2}{4} \right)^{n_1}.$$

Therefore

$$\int_0^\infty x^p \cos(ax+b)\, dx = \int_0^\infty x^p \sum_{n_1=0}^\infty \phi_{n_1} \frac{\Gamma(\frac{1}{2})}{4^{n_1}\,\Gamma(n_1+\frac{1}{2})} (ax+b)^{2n_1}\, dx. \tag{7.3}$$

The bracket expansion

$$(ax+b)^{2n_1} = \sum_{n_2=0}^\infty \sum_{n_3=0}^\infty \phi_{2,3} a^{n_2} b^{n_3} x^{n_2} \frac{\langle -2n_1 + n_2 + n_3 \rangle}{\Gamma(-2n_1)}, \tag{7.4}$$

gives

$$\int_0^\infty x^p \cos(ax+b)\,dx = \tag{7.5}$$
$$\sum_{n_1,n_2,n_3 \geqslant 0} \phi_{1,2,3} \frac{\Gamma(\frac{1}{2})\, a^{n_2} b^{n_3}}{4^{n_1}\Gamma(n_1+\frac{1}{2})\,\Gamma(-2n_1)} \langle -2n_1+n_2+n_3\rangle \langle n_2+p+1\rangle.$$

The vanishing of the bracket $\langle n_2+p+1\rangle$ determines $n_2^* = -p-1$. There is one sum and two possible choices for a free index.

Case 1: n_1 is free. Then $n_3^* = 2n_1-p-1$ and the corresponding determinant is -1. The contribution to the integral is given by

$$\begin{aligned} S_1 &= \sum_{n_1 \geqslant 0} \phi_{n_1} \frac{1}{|-1|} \frac{\Gamma(\frac{1}{2}) a^{n_2^*} b^{n_3^*}}{4^{n_1}\Gamma(n_1+\frac{1}{2})\,\Gamma(-2n_1)} \Gamma(-n_2^*)\Gamma(-n_3^*) \\ &= \sum_{n_1 \geqslant 0} \phi_{n_1} \frac{\Gamma(\frac{1}{2}) a^{-p-1} b^{2n_1+p+1}}{4^{n_1}\Gamma(n_1+\frac{1}{2})\,\Gamma(-2n_1)} \Gamma(p+1)\Gamma(-2n_1-p-1). \end{aligned}$$

Each term $1/\Gamma(-2n_1)$ vanishes, it follows that $S_1 = 0$.

Case 2: n_3 is free. Then $n_1^* = \frac{1}{2}(n_3-p-1)$ and the contribution to the integral is

$$\begin{aligned} S_2 &= \sum_{n_3 \geqslant 0} \phi_{n_3} \frac{1}{2} \frac{\Gamma(\frac{1}{2}) a^{n_2^*} b^{n_3}}{4^{n_1^*}\Gamma(n_1^*+\frac{1}{2})\,\Gamma(-2n_1^*)} \Gamma(-n_1^*)\Gamma(-n_2^*) \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)2^p}{a^{p+1}} \sum_{n_3=0}^{\infty} \phi_{n_3} \frac{b^{n_3}\Gamma(\frac{1}{2}(-n_3+p+1)).}{2^{n_3}\Gamma(\frac{1}{2}(n_3-p))\,\Gamma(-n_3+p+1)}. \end{aligned}$$

The factors in the last summand can be simplified to produce

$$\begin{aligned} S_2 &= \frac{\Gamma(p+1)}{a^{p+1}} \left[-\sin\frac{\pi p}{2} \sum_{k=0}^{\infty} \frac{(-b^2)^k}{\Gamma(2k+1)} - b\cos\frac{\pi p}{2} \sum_{k=0}^{\infty} \frac{(-b^2)^k}{\Gamma(2k+2)} \right] \\ &= \frac{\Gamma(p+1)}{a^{p+1}} \left[-\sin\frac{\pi p}{2}\cos b - b\cos\frac{\pi p}{2}\sin b \right] \\ &= -\frac{\Gamma(p+1)}{a^{p+1}} \sin\left(\frac{\pi p}{2}+b\right) \end{aligned}$$

Adding all the finite contributions of free indices gives the evaluation

$$\int_0^\infty x^p \cos(ax+b)\,dx = -\frac{\Gamma(p+1)}{a^{p+1}} \sin\left(\frac{\pi p}{2}+b\right). \tag{7.6}$$

8. The introduction of a parameter

This section illustrates the evaluation of entry **3.249**

$$\int_0^\infty \left[e^{-x} - (1+x)^{-\mu}\right] \frac{dx}{x} = \psi(\mu), \quad \text{for } \operatorname{Re}\mu > 0. \tag{8.1}$$

A classical evaluation of this entry appears in [**18**].

To apply the method of brackets, consider first the integral

$$I(\varepsilon) = \int_0^\infty \frac{\exp(-x) - (1+x)^{-\mu}}{x^{1-\varepsilon}}\, dx. \tag{8.2}$$

The result is obtained by letting $\varepsilon \to 0$. Now compute the bracket series associated to the integrand in (8.2) to obtain

$$I(\varepsilon) = \sum_{k \geqslant 0} \phi_k \left[1 - \frac{\Gamma(\mu+n)}{\Gamma(\mu)} \right] \langle k + \varepsilon \rangle. \tag{8.3}$$

Therefore,

$$I(\varepsilon) = \Gamma(\varepsilon) \left[1 - \frac{\Gamma(\mu-\varepsilon)}{\Gamma(\mu)} \right]. \tag{8.4}$$

To obtain the value of (8.1), simply use the expansion

$$\Gamma(\varepsilon) \left[1 - \frac{\Gamma(\mu-\varepsilon)}{\Gamma(\mu)} \right] = \psi(\mu) - \left(\frac{\psi'(\mu)}{2} - \frac{1}{2}\psi^2(\mu) + \gamma\psi(\mu) \right) \varepsilon + O(\varepsilon^2)$$

as $\varepsilon \to 0$.

Conclusions. The examples given here, all taken from the classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik, have been evaluated using the method of brackets. This illustrate the great flexibility of this method. The rules for evaluation have been partially justified via Ramanujan's Master Theorem.

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