Bernoulli Polynomials

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The evaluation of some finite sums

The sums

$$S_1(n) := 1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2} \tag{1}$$

and

$$S_2(n) := 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

can be easily verified by induction.

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$$S_a(n) = \sum_{k=1}^{n-1} k^a. \tag{3}$$

Theorem

The sum $S_a(n)$ is a polynomial in n of degree $a+1$.

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The sum $S_a(n)$ is a polynomial in n of degree $a+1$.

The evaluation of some finite sums: the proof

Proof.

The identity

$$(k+1)^{a+1} - k^{a+1} = \sum_{j=0}^{a} \binom{a+1}{j} k^j \tag{4}$$

is summed for $k = 1$ to $k = n-1$ to produce

$$n^{a+1} - 1 = \sum_{j=0}^{a} \binom{a+1}{j} S_j(n), \tag{5}$$

so that

$$\binom{a+1}{a} S_a(n) = n^{a+1} - 1 - \sum_{j=0}^{a-1} \binom{a+1}{j} S_j(n). \tag{6}$$

The result now follows by induction. □

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Examples

Starting with the value

$$S_0(n) = \sum_{k=1}^{n-1} 1 = n - 1, \tag{7}$$

the recurrences give

$$\begin{aligned} S_1(n) &= \frac{n^2}{2} - \frac{n}{2}, \\ S_2(n) &= \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}, \\ S_3(n) &= \frac{n^4}{4} - \frac{n^3}{2} + \frac{n^2}{4}, \end{aligned}$$

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Use the value $S_0(n) = n - 1$ and the recurrence to prove that $S_a(0) = 0$ for $a \geq 1$.

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Bernoulli polynomials

Definition

For $a \geq 2$ introduce the polynomial $B_a(n)$ by the identity

$$S_{a-1}(n) = \sum_{k=0}^{n-1} k^{a-1} = \frac{B_a(n) - B_a}{a}. \tag{8}$$

The constant B_a (undefined for now) satisfies

$$B_a = B_a(0). \tag{9}$$

This agrees with the Exercise and it also consistent with the usual convention that empty sums are zero. The polynomial $B_a(n)$ is called the **Bernoulli polynomial** and B_a is the **Bernoulli number**.

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Bernoulli polynomials: continued

In the exceptional case $a = 1$ we define

$$S_{a-1}(n) = \sum_{k=0}^{n-1} k^{a-1} = \frac{B_a(n) + B_a}{a}. \tag{10}$$

The value $S_0(n) = n - 1$ and the value $B_1 = -\frac{1}{2}$ determined later, show that $B_1(n) = n - \frac{1}{2}$.

Exercise

Check that $B_2(n) = n^2 - n + B_2$ and $B_3(n) = n^3 - \frac{3}{2}n^2 + \frac{1}{2}n + B_3$.

The Bernoulli numbers B_a will be determined in later. It will follow that $B_2 = \frac{1}{6}$ and $B_3 = 0$.

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The **Bernoulli polynomials** $B_a(x)$ are defined by

$$B_a(n) = B_a + a\sum_{k=1}^{n-1} k^{a-1}. \tag{11}$$

Exercise

Check that the Bernoulli polynomial satisfies the identity

$$B_a(x+1) = B_a(x) + ax^{a-1}. \tag{12}$$

Conclude that, for $a > 1$, we have $B_a(1) = B_a$.

The identity (12) and the values of $B_a(x)$ for $0 \leq x \leq 1$ determine $B_a(x)$ for all $x \in \mathbb{R}$.

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$$B_{a+1}(n) = B_{a+1} + n^{a+1} - n - \sum_{j=1}^{a-1} \frac{\binom{a+1}{j}}{j+1} \left[B_{j+1}(n) - B_{j+1} \right]. \quad (13)$$

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The exponential generating function

Definition

The **Bernoulli numbers** B_a are defined by

$$\frac{t}{e^t - 1} = \sum_{a=0}^{\infty} B_a \frac{t^a}{a!}. \tag{14}$$

Theorem

The exponential generating function of the Bernoulli polynomials is given by

$$\sum_{a=0}^{\infty} B_a(x) \frac{t^a}{a!} = \frac{te^{xt}}{e^t - 1}. \tag{15}$$

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The exponential generating function: the proof

Proof.

Start with

$$\begin{aligned}
\sum_{a=0}^{\infty}\left[B_a(x)-B_a\right]\frac{t^a}{a!} &= \sum_{a=1}^{\infty}\frac{t^a}{(a-1)!}\sum_{k=1}^{x-1}k^{a-1} \\
&= t\sum_{k=1}^{x-1}\sum_{a=0}^{\infty}\frac{(tk)^a}{a!} \\
&= t\cdot\frac{e^{xt}-1}{e^t-1}.
\end{aligned}$$

□

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\end{aligned}$$

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Elementary properties of Bernoulli numbers

Exercise

Use the gf to check that $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_2 = \frac{1}{6}$.

More efficient method:

Proposition

The Bernoulli numbers satisfy

$$\sum_{k=1}^{a} \binom{a}{k} B_{a-k} = 0. \tag{16}$$

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A recurrence: the proof

The generating function for B_a can be expressed as

$$\begin{aligned} t &= (e^t - 1) \times \sum_{a=0}^{\infty} B_a \frac{t^a}{a!} \\ &= \sum_{k=1}^{\infty} \frac{t^k}{k!} \times \sum_{a=0}^{\infty} B_a \frac{t^a}{a!} \\ &= \sum_{k,a} \frac{B_a}{k!\,a!} t^{a+k}. \end{aligned}$$

Now let $r = a + k$ and eliminate the index a to obtain

$$t = \sum_{r=1}^{\infty} \left[\sum_{k=1}^{r} \frac{B_{r-k}}{k!\,(r-k)!} \right] t^r. \tag{17}$$

The result follows by matching equal powers.

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Bernoulli numbers are rational

Corollary

The Bernoulli numbers B_a are rational numbers.

Proof.

The identity (16) can be written as

$$B_a = -\frac{1}{a+1}\sum_{j=0}^{a-1}\binom{a+1}{j}B_j. \tag{18}$$

The result follows by induction on a. $\square$

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Some exercises

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Use the recurrence to confirm the first few values of B_a. The numerators can get very large, for example

$$B_{38} = \frac{2929993913841559}{6} \text{ and } B_{40} = -\frac{261082718496449122051}{13530}.$$

Exercise

Make a list of the denominators of the even Bernoulli numbers. This is how it starts

$$1, 6, 30, 42, 30, 66, 2730, 6, 510, 798, 330, 138, 2730, 6, 870.$$

The fact that the denominator of B_a is often 6 will be explained later. The first values for which this happens are

$$2, 14, 26, 34, 38, 62, 74, 86, 94, 98.$$

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Vanishing of B_{2a+1}.

Proposition

The value $B_1 = -\frac{1}{2}$ is the only non-zero Bernoulli number with an odd index.

Proof.

The function

$$\frac{t}{e^t - 1} - 1 + \frac{t}{2} = \frac{t}{2}\left[\frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}}\right] - 1 \tag{19}$$

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The signs of B_{2a}.

This proof is due to L. J. Mordell.

Theorem

The Bernoulli numbers B_{2a} satisfy $(-1)^{a-1}B_{2a} > 0$.

First an intermediate result.

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The Bernoulli numbers satisfy

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The signs of B_{2a}: the proof.

Write

$$\frac{t}{e^t - 1} = \sum_{a=0}^{\infty} B_a \frac{t^a}{a!} \tag{21}$$

so that $B_0 = 1$, $B_1 = -1/2$ and $B_{2a+1} = 0$ for $a > 1$. Then

$$\frac{t}{e^t + 1} = \frac{t}{e^t - 1} - \frac{2t}{e^{2t} - 1} = -\sum_{a=0}^{\infty} (2^a - 1) B_a \frac{t^a}{a!}$$

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The signs of B_{2a}: the proof (continuation).

Multiply by $t/(e^t - 1)$ so the left hand side becomes $t^2/(e^{2t} - 1)$ and expanding we obtain

$$\frac{t}{2}\sum_{a=0}^{\infty} B_a 2^a \frac{t^a}{a!} \quad = \quad -\left(\sum_{r=0}^{\infty}(2^r - 1)B_r\frac{t^r}{r!}\right) \times \left(\sum_{s=0}^{\infty} B_s \frac{t^s}{s!}\right).$$

Now equate the coefficients of t^{2a} to prove the intermediate result.

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The signs of B_{2a}: proof of the main result.

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Define $b_a = (-1)^{a-1} B_{2a}$. Then (20) becomes

$$b_a = \sum_{r=1}^{a-1} \frac{2^{2r}-1}{2^{2a}-1} \binom{2a}{2r} b_r b_{a-r}. \quad (22)$$

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Differential properties

Proposition

The Bernoulli polynomials $B_a(x)$ satisfy

$$B_a'(x) = aB_{a-1}(x) \tag{23}$$

Proof.

Differentiate the generating function and use $B_0(x) = 1$, to obtain

$$\frac{te^{xt}}{e^t - 1} = \sum_{a=1}^{\infty} B_a'(x) \frac{t^{a-1}}{a!}. \tag{24}$$

The result now follows from the generating function. □

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Some exercises

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Use the generating function to prove

$$B_a(x+1) = B_a(x) + ax^{a-1}. \tag{25}$$

This identity follows directly from the relation

$$B_a(x) = B_a(0) + a\sum_{k=0}^{x-1} k^{a-1}. \tag{26}$$

for $x \in \mathbb{N}$.

Exercise

Check that $B_a(1) = B_a(0)$ *for* $a \geq 2$. *Compare these values for* $a = 0$ *and* $a = 1$.

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Check the special value

$$B_a\left(\tfrac{1}{2}\right) = -(1-2^{1-a})B_a. \tag{27}$$

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Use the generating function to prove that

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The map

$$\mathfrak{D}(P(x)) = P(x+1) - P(x) \tag{29}$$

acts on the space of polynomials. The polynomial $g_a(x) = B_a(x)/a$ *satisfies*

$$\mathfrak{D}(g_a) = x^{a-1} \tag{30}$$

Study the operator $\mathfrak{D}$ *from the point of view of Linear Algebra.*

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$B_a(x)$ in terms of B_a

Now we write $B_a(x)$ only using the Bernoulli numbers.

Theorem

The Bernoulli polynomials are given by

$$B_a(x) = \sum_{j=0}^{a} \binom{a}{j} B_j x^{a-j}. \tag{31}$$

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$B_a(x)$ in terms of B_a: the proof

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Start with the generating function

$$\begin{aligned}
\sum_{a=0}^{\infty} B_a(x)\frac{t^a}{a!} &= \frac{te^{xt}}{e^t-1} = e^{xt} \times \frac{t}{e^t-1} \\
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&= \sum_{a=0}^{\infty} \left[\sum_{j=0}^{a} \binom{a}{j} B_j x^{a-j}\right] \frac{t^a}{a!}.
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This is the stated formula.

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The inversion formula

The Bernoulli polynomials $\{B_a(x) : 0 \leq a \leq n\}$ form a basis for the vector space of polynomials of degree less or equal than n. In particular, the polynomial x^j $(0 \leq j \leq n)$ is a linear combination of B_a. The next exercise gives the explicit values of the coefficients.

Exercise

Prove the inversion formula

$$x^n = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} B_j(x). \tag{32}$$

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Exercise

Prove the duplication formula

$$B_a(2x) = 2^{a-1}\left[B_a(x) + B_a(x+\tfrac{1}{2})\right]. \tag{34}$$

This is similar to the trigonometric identity

$$\cot 2x = \cot x + \cot(x+\tfrac{1}{2}) \tag{35}$$

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Sequences related to Bernoulli numbers

Exercise

Write the expansion of the cotangent function in terms of Bernoulli numbers.

Exercise

The tangent numbers T_n are defined by the expansion

$$\tan x = \sum_{n=0}^{\infty} T_n \frac{x^n}{n!}. \tag{37}$$

Use the identity

$$\cot x - 2\cot 2x = \tan x \tag{38}$$

to produce

$$(-1)^{n-1}(2^{2n} - 1)B_{2n} = 2nT_{2n-1}. \tag{39}$$

Conclude that $(-1)^{n-1}B_{2n} > 0$. This proof is due to Carlitz.

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Some exercises: continuation

Exercise

There are many series of elementary functions that can be expressed in terms of Bernoulli numbers. Use the series for $\tan x$ *to establish*

$$\cot x = \sum_{a=0}^{\infty} (-1)^a \frac{2^{2a}}{(2a)!} B_{2a} x^{2a-1} \tag{40}$$

$$\operatorname{cosec} x = \sum_{a=0}^{\infty} (-1)^{a-1} \frac{(2^{2a}-2)}{(2a)!} B_{2a} x^{2a-1}. \tag{41}$$

This explains why Calculus courses discuss the Taylor series of $\sin x$ *and* $\cos x$*, but not the other four trigonometric functions. Observe that the series for* $\sec x$ *is not in the list.*

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$$\operatorname{cosec} x = \sum_{a=0}^{\infty} (-1)^{a-1} \frac{(2^{2a}-2)}{(2a)!} B_{2a} x^{2a-1}. \tag{41}$$

This explains why Calculus courses discuss the Taylor series of $\sin x$ *and* $\cos x$*, but not the other four trigonometric functions. Observe that the series for* $\sec x$ *is not in the list.*

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Integrals involving Bernoulli polynomials

The first examples follow from

$$B'_{a+1}(x) = (a+1)B_a(x), \quad a \geq 1, \tag{43}$$

Let $a \geq 1$. Then

$$\int_0^1 B_a(x)\,dx = 0. \tag{44}$$

Proof.

Integrating (43) yields

$$B_{a+1}(x_1) - B_{a+1}(x_2) = (a+1)\int_{x_1}^{x_2} B_a(x)\,dx. \tag{45}$$

Now take $x_1 = 1$ and $x_2 = 0$ and use $B_a(1) = B_a(0) = B_a$ to obtain the result. $\square$

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Integrals of Bernoulli polynomials can be used to generate some identities for Bernoulli numbers. For example,

$$\begin{aligned} \int_0^1 x B_a(x)\,dx &= \sum_{j=0}^{a} \binom{a}{j} B_j \int_0^1 x^{a-j+1}\,dx \\ &= \sum_{j=0}^{a} \binom{a}{j} \frac{B_j}{a-j+2}. \end{aligned}$$

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The product of two Bernoulli polynomials is given by

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Products of Bernoulli polynomials: the proof

The set of polynomials

$$\mathfrak{P}_n = \{B_0(x), B_1(x), B_2(x), \cdots, B_n(x)\} \tag{47}$$

forms a basis for the vector space of polynomials of degree at most n. It follows that there exists numbers such that

$$B_p(x)B_q(x) = \sum_{j=0}^{p+q} \alpha_j(p,q)B_j(x). \tag{48}$$

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We begin with the identity

$$\frac{uv}{(e^u-1)(e^v-1)}\times\frac{e^{u+v}-1}{u+v}=\frac{uv}{u+v}+\frac{v}{u+v}\frac{u}{e^u-1}+\frac{u}{u+v}\frac{v}{e^v-1},$$

and we write the right hand side as

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$$B_{n_1}(x)B_{n_2}(x) = \sum_{k=0}^{k(n_1,n_2)} \left[n_1\binom{n_2}{2k} + n_2\binom{n_1}{2k} \right] \frac{B_{2k}}{n_1+n_2-2k} B_{n_1+n_2-2k}(x) + (-1)^{n_1+1}\frac{n_1!\,n_2!}{(n_1+n_2)!}B_{n_1+n_2}, \quad (49)$$

where $k(n_1,n_2) = \mathrm{Max}\{\lfloor n_1/2 \rfloor, \lfloor n_2/2 \rfloor\}$.
In theory, (49) yields expressions for a product of many number Bernoulli polynomials.

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$$B_{n_1}(x)B_{n_2}(x) = \sum_{k=0}^{k(n_1,n_2)} \left[n_1 \binom{n_2}{2k} + n_2 \binom{n_1}{2k} \right] \frac{B_{2k}}{n_1 + n_2 - 2k} B_{n_1+n_2-2k}(x) + (-1)^{n_1+1} \frac{n_1!\, n_2!}{(n_1+n_2)!} B_{n_1+n_2}, \quad (49)$$

where $k(n_1, n_2) = \mathrm{Max}\{\lfloor n_1/2 \rfloor, \lfloor n_2/2 \rfloor\}$.
In theory, (49) yields expressions for a product of many number Bernoulli polynomials.

Products of Bernoulli polynomials (continued)

$$B_n^2(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k} B_{2n-2k}(x) + (-1)^{n+1} \frac{B_{2n}}{\binom{2n}{n}}, \qquad (50)$$

and

$$B_n^3(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k} \times$$

$$\sum_{j=0}^{n-k} \left[n\binom{2n-2k}{2j} + 2(n-k)\binom{n}{2j} \right] \frac{B_{2j} B_{3n-2k-2j}(x)}{3n-2k-2j}$$

$$+ (-1)^{n+1} \left[\binom{2n}{n}^{-1} B_{2n} B_n(x) \frac{B_{2n}}{\binom{2n}{n}} B_n(x) + 2n!^3 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n-2k-1}{n-1} \right.$$

Products of Bernoulli polynomials (continued)

$$B_n^2(x) \;=\; \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k} B_{2n-2k}(x) + (-1)^{n+1} \frac{B_{2n}}{\binom{2n}{n}}, \qquad (50)$$

and

$$B_n^3(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k} \times$$

$$\sum_{j=0}^{n-k} \left[n\binom{2n-2k}{2j} + 2(n-k)\binom{n}{2j} \right] \frac{B_{2j} B_{3n-2k-2j}(x)}{3n-2k-2j}$$

$$+ (-1)^{n+1} \left[\binom{2n}{n}^{-1} B_{2n} B_n(x) \frac{B_{2n}}{\binom{2n}{n}} B_n(x) + 2n!^3 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n-2k-1}{n-1} \frac{B}{(2k} \right.$$

$$B_n^2(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k} B_{2n-2k}(x) + (-1)^{n+1} \frac{B_{2n}}{\binom{2n}{n}}, \qquad (50)$$

and

$$B_n^3(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k} \times$$
$$\sum_{j=0}^{n-k} \left[n\binom{2n-2k}{2j} + 2(n-k)\binom{n}{2j} \right] \frac{B_{2j} B_{3n-2k-2j}(x)}{3n-2k-2j}$$
$$+ (-1)^{n+1} \left[\binom{2n}{n}^{-1} B_{2n} B_n(x) \frac{B_{2n}}{\binom{2n}{n}} B_n(x) + 2n!^3 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n-2k-1}{n-1} \frac{B}{(2k}\right.$$

Products of Bernoulli polynomials (continued)

$$B_n^2(x) \quad = \quad \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k} B_{2n-2k}(x) + (-1)^{n+1} \frac{B_{2n}}{\binom{2n}{n}}, \qquad (50)$$

and

$$B_n^3(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k} \times$$

$$\sum_{j=0}^{n-k} \left[n\binom{2n-2k}{2j} + 2(n-k)\binom{n}{2j} \right] \frac{B_{2j} B_{3n-2k-2j}(x)}{3n-2k-2j}$$

$$+ (-1)^{n+1} \left[\binom{2n}{n}^{-1} B_{2n} B_n(x) \frac{B_{2n}}{\binom{2n}{n}} B_n(x) + 2n!^3 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n-2k-1}{n-1} \frac{B}{(2k} \right.$$

Products of Bernoulli polynomials (continued)

$$B_n^2(x) \quad = \quad \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k} B_{2n-2k}(x) + (-1)^{n+1} \frac{B_{2n}}{\binom{2n}{n}}, \quad (50)$$

and

$$B_n^3(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k} \times$$

$$\sum_{j=0}^{n-k} \left[n\binom{2n-2k}{2j} + 2(n-k)\binom{n}{2j} \right] \frac{B_{2j} B_{3n-2k-2j}(x)}{3n-2k-2j}$$

$$+ (-1)^{n+1} \left[\binom{2n}{n}^{-1} B_{2n} B_n(x) \frac{B_{2n}}{\binom{2n}{n}} B_n(x) + 2n!^3 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n-2k-1}{n-1} \frac{B}{(2k} \right.$$