THE BERNOULLI NUMBERS

The **Bernoulli numbers** are defined here by the exponential generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \tag{1}$$

The first one is easy to compute:

$$\begin{aligned} B_0 &= \lim_{t \to 0} \frac{t}{e^t - 1} \\ &= \lim_{t \to 0} \frac{1}{e^t} \\ &= 1, \end{aligned} \tag{2}$$

and

$$\begin{aligned} B_1 &= \lim_{t \to 0} \frac{d}{dt} \left(\frac{t}{e^t - 1} \right) \\ &= \lim_{t \to 0} \frac{-1 + e^t - te^t}{(e^t - 1)^2} \\ &= \lim_{t \to 0} \frac{-t}{2(e^t - 1)} \\ &= -\lim_{t \to 0} \frac{1}{2e^t} \\ &= -\tfrac{1}{2}. \end{aligned} \tag{3}$$

Ideally one would like to obtain a recurrence for these numbers. The only tool we have is the ordinary generating function, so we work with it. The relation (1) is written as

$$t = \left(e^t - 1 \right) \times \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \right) \tag{4}$$

that can be written as

$$t = \left(\sum_{j=1}^{\infty} \frac{t^j}{j!} \right) \times \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \right) \tag{5}$$

Dividing by t we can write the series on the left as

$$\sum_{j=1}^{\infty} \frac{t^{j-1}}{j!} = \sum_{j=0}^{\infty} \frac{t^j}{(j+1)!} \tag{6}$$

and (5) becomes

$$1 = \left(\sum_{j=0}^{\infty} \frac{t^j}{(j+1)!} \right) \times \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \right) \tag{7}$$

How does one multiply series. In order to simplify (7) we will obtain an expression for the product of two power series:

$$\left(\sum_{j=0}^{\infty} a_j t^j\right) \times \left(\sum_{k=0}^{\infty} b_k t^k\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_j b_k t^{j+k}. \tag{8}$$

The double sum on the right corresponds to summing over all point on the first quadrant $\mathbb{N}_0 \times \mathbb{N}_0$. The same set of indices can be covered by lines of slope r, that is, summing over all indices (i,j) with $r = i+j$ fixed and then summing over all values of $r \in \mathbb{N}_0$. This gives

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_j b_k t^{j+k} = \sum_{r=0}^{\infty}\sum_{j+k=r} a_j b_k t^{j+k}. \tag{9}$$

In the inner sum the value of the index r is fixed and we now eliminate the index j, to obtain $j = r-k$ with the range $0 \leq k \leq r$. This gives

$$\sum_{r=0}^{\infty}\sum_{i+j=r} a_j b_k t^{j+k} = \sum_{r=0}^{\infty}\sum_{k=0}^{r} a_{r-k} b_k t^r. \tag{10}$$

The conclusion is that

$$\left(\sum_{j=0}^{\infty} a_j t^j\right) \times \left(\sum_{k=0}^{\infty} b_k t^k\right) = \sum_{r=0}^{\infty}\left(\sum_{k=0}^{r} a_{r-k} b_k\right) t^r. \tag{11}$$

This can also be written in the following form:

$$\begin{aligned}\left(\sum_{j=0}^{\infty} \frac{a_j}{j!} t^j\right) \times \left(\sum_{k=0}^{\infty} \frac{b_k}{k!} t^k\right) &= \sum_{r=0}^{\infty}\left(\sum_{k=0}^{r} \frac{a_{r-k}}{(r-k)!}\frac{b_k}{k!}\right) t^r \\ &= \sum_{r=0}^{\infty}\left(\sum_{k=0}^{r} \binom{r}{k} a_{r-k} b_k\right) \frac{t^r}{r!}\end{aligned} \tag{12}$$

Theorem 1. *The coefficient of t^r in the product*

$$\left(\sum_{j=0}^{\infty} a_j t^j\right) \times \left(\sum_{k=0}^{\infty} b_k t^k\right) \tag{13}$$

is

$$\sum_{k=0}^{r} a_{r-k} b_k = \sum_{k=0}^{r} a_k b_{r-k}. \tag{14}$$

The coefficient of $t^r/r!$ in the product

$$\left(\sum_{j=0}^{\infty} \frac{a_j}{j!} t^j\right) \times \left(\sum_{k=0}^{\infty} \frac{b_k}{k!} t^k\right) \tag{15}$$

is

$$\sum_{k=0}^{r} \binom{r}{k} a_{r-k} b_k = \sum_{k=0}^{r} \binom{r}{k} a_k b_{r-k}. \tag{16}$$

Now apply the rule in (16) to the identity (7) written in the form

$$1 = \left(\sum_{j=0}^{\infty} \frac{1}{j+1} \frac{t^j}{j!} \right) \times \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \right) \tag{17}$$

to obtain

$$\sum_{k=0}^{r} \binom{r}{k} \frac{B_k}{r-k+1} = \begin{cases} 0 & \text{if } r \neq 0 \\ 1 & \text{if } r = 0 \end{cases} \tag{18}$$

Theorem 2. *The Bernoulli numbers satisfy the recurrence*

$$B_r = -\sum_{k=0}^{r-1} \binom{r}{k} \frac{B_k}{r-k+1}, \ \textit{for } r > 0. \tag{19}$$

Proof. Solve the relation (18) for B_r. □

Corollary 3. *The Bernoulli numbers are rational numbers.*

This recurrence can be used to generate the sequence of Bernoulli numbers. The first few are

$$\left\{ 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, 0, \frac{7}{6} \right\} \tag{20}$$

and it can be seen from this table that, aside from $B_1 = -\frac{1}{2}$ the Bernoulli numbers with odd index vanish. This must not be hard to prove.

Consider the generating function

$$G(t) = \frac{t}{e^t - 1} \tag{21}$$

and modify it to eliminate the term corresponding to B_1. That is, define

$$G_1(t) = \frac{t}{e^t - 1} + \frac{t}{2}. \tag{22}$$

This can be reduced to

$$\begin{aligned} G_1(t) &= t \left[\frac{1}{e^t - 1} + \frac{1}{2} \right] \\ &= \frac{t}{2} \cdot \frac{e^t + 1}{e^t - 1} \end{aligned} \tag{23}$$

and the second factor is

$$\frac{e^t + 1}{e^t - 1} = \frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}} \tag{24}$$

and its is clear that this is an odd function. Therefore, becuase of the extra factor $t/2$, the function $G_1(t)$ is an even function. As such it has only even terms in its generating function expansion.

Theorem 4. *For n odd and $n \geq 3$, the Bernoulli number B_n vanishes. That is*

$$B_{2n+1} = 0, \ \textit{for } n \geq 1. \tag{25}$$

With this result, the generating function (1) for the Bernoulli numbers can be written as

$$\begin{aligned} \frac{t}{e^t-1} &= \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \\ &= 1 - \frac{t}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k}}{(2k)!}. \end{aligned} \tag{26}$$

Many properties of the Bernoulli numbers are established by clever manipulations of the generating function. The details given next appear in a paper by L. J. Mordell in the American Mathematical Monthly, volume 80, 1973, pages 547-548.

Start with the identity

$$\frac{t}{e^t+1} = \frac{t}{e^t-1} - \frac{2t}{e^{2t}-1} \tag{27}$$

and expand both sides in series. The right-hand side is easy:

$$\begin{aligned} \frac{t}{e^t-1} - \frac{2t}{e^{2t}-1} &= \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} - \sum_{k=0}^{\infty} B_k \frac{2^k t^k}{k!} \\ &= \sum_{k=0}^{\infty} (1-2^k) B_k \frac{t^k}{k!} \\ &= -\sum_{k=1}^{\infty} (2^k-1) B_k \frac{t^k}{k!}. \end{aligned} \tag{28}$$

The expansion of the left-hand side is not so obvious, but the clever idea is to multiply by a nice factor. Indeed,

$$\frac{t}{e^t+1} \times \frac{t}{e^t-1} = \frac{t^2}{e^{2t}-1} \tag{29}$$

and the right-hand side can be written as

$$\frac{t^2}{e^{2t}-1} = \frac{t}{2} \cdot \frac{2t}{e^{2t}-1} \tag{30}$$

and this can be expanded as

$$\frac{t^2}{e^{2t}-1} = \frac{t}{2} \cdot \frac{2t}{e^{2t}-1} = \frac{t}{2} \cdot \sum_{k=0}^{\infty} B_k \frac{2^k t^k}{k!}. \tag{31}$$

Now take the identity (27) and multiply it by $t/(e^t-1)$ to produce

$$\frac{t}{2} \cdot \frac{2t}{e^{2t}-1} = \left(\frac{t}{e^t-1} \right) \times \left(\frac{t}{e^t-1} - \frac{2t}{e^{2t}-1} \right) \tag{32}$$

Expanding in series gives

$$\frac{t}{2} \cdot \sum_{k=0}^{\infty} B_k \frac{2^k t^k}{k!} = \left(\sum_{j=0}^{\infty} B_j \frac{t^j}{j!} \right) \times \left[-\sum_{i=1}^{\infty} (2^i-1) B_i \frac{t^i}{i!} \right] \tag{33}$$

The product on the right hand the form in (16) with

$$a_j = B_j \text{ and } b_k = -(2^k-1)B_k. \tag{34}$$

Therefore, the coefficient of $t^r/r!$ for the product on the right is given by

$$-\sum_{i=0}^{r}\binom{r}{i}(2^i-1)B_iB_{r-i} \tag{35}$$

and on the left-hand side this coefficient is

$$B_{r-1}2^{r-2}. \tag{36}$$

Therefore, if r is even, say $r=2s$ with $s>1$ the left-hand side is 0 and we have

$$\sum_{i=0}^{2s}\binom{2s}{i}(2^i-1)B_iB_{2s-i}=0. \tag{37}$$

The term for $i=0$ vanishes, the term for $i=1$ also vansihes because of the factor B_{2s-1} (this is an odd index for Bernoulli number and $2s-1>1$). Therefore the sum starts at $i=2$ and it must contain only even indices i, because $B_i=0$ for i odd. Let $i=2j$ and write (37) as

$$\sum_{j=1}^{s}\binom{2s}{2j}(2^{2j}-1)B_{2j}B_{2s-2j}=0. \tag{38}$$

The summand for $j=s$ is

$$(2^{2s}-1)B_{2s} \tag{39}$$

and if we solve for it leads to

$$B_{2s}=-\sum_{j=1}^{s-1}\frac{2^{2j}-1}{2^{2s}-1}\binom{2s}{2j}B_{2j}B_{2s-2j}. \tag{40}$$

This is a recurrence for the Bernoulli numbers that involve only even indices.

Theorem 5. *The Bernoulli numbers of even index satisfy the recurrence*

$$B_{2s}=-\frac{1}{2^{2s}-1}\sum_{j=1}^{s-1}(2^{2j}-1)\binom{2s}{2j}B_{2j}B_{2s-2j} \tag{41}$$

with initial condition $B_0=1$.

Another observation coming from the list in (20) is that the sign of the non-zero Bernoulli numbers alternate. This is easy to prove from the recurrence (41).

Corollary 6. *For* $n\in\mathbb{N}$

$$(-1)^{n-1}B_{2n}>0. \tag{42}$$

Proof. Define

$$b_n=(-1)^{n-1}B_{2n} \tag{43}$$

and replace in (41) to obtain

$$b_n=\sum_{j=1}^{n-1}\frac{2^{2j}-1}{2^{2n}-1}\binom{2n}{2j}b_jb_{n-j}. \tag{44}$$

The initial condition $b_1=\frac{1}{6}$ shows that $b_n>0$ for all $n\in\mathbb{N}$. $\square$

A second identity comes by manipulations of the generating function. The **Bernoulli numbers** have been defined here by the exponential generating in (1) by

$$f(t) = \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \tag{45}$$

The identity comes by differentiating the generating function to obtain

$$\frac{d}{dt}\frac{t}{e^t - 1} = \frac{1}{e^t - 1} - \frac{te^t}{(e^t-1)^2} \tag{46}$$

to produce

$$f'(t) = \frac{f(t)}{t} - f(t) - \frac{f^2(t)}{t}. \tag{47}$$

This is written as

$$\sum_{k=0}^{\infty} kB_k \frac{t^{k-1}}{k!} = \sum_{k=0}^{\infty} B_k \frac{t^{k-1}}{k!} - \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} - \sum_{r=0}^{\infty} \left(\sum_{j=0}^{r} \binom{r}{j} B_j B_{r-j} \right) \frac{t^r}{r!} \tag{48}$$

Now compare the coefficients of t^r to produce

$$\frac{B_{r+1}}{r!} = \frac{B_{r+1}}{(r+1)!} - \frac{B_r}{r!} - \frac{1}{(r+1)!} \sum_{j=0}^{r+1} \binom{r+1}{j} B_j B_{r+1-j} \tag{49}$$

that can be written as

$$rB_{r+1} = -(r+1)B_r - \sum_{j=0}^{r+1} \binom{r+1}{j} B_j B_{r+1-j} \tag{50}$$

and this leads to

$$(r+2)B_{r+1} = -(r+1)B_r - \sum_{j=1}^{r} \binom{r+1}{j} B_j B_{r+1-j}. \tag{51}$$

Theorem 7. *The Bernoulli numbers satisfy the recurrence*

$$(r+2)B_{r+1} = -(r+1)B_r - \sum_{j=1}^{r} \binom{r+1}{j} B_j B_{r+1-j} \tag{52}$$

for $r \geq 0$.

Take $r = 2u - 1$ to be odd, then (52) gives

$$(2u+1)B_{2u} = -\sum_{j=1}^{2u-1} \binom{2u}{j} B_j B_{2u-j} \tag{53}$$

and it follows that in the sum we should only consider even indices, to produce

Theorem 8. *The Bernoulli numbers satisfy the recurrence*

$$(2u+1)B_{2u} = -\sum_{r=1}^{u-1} \binom{2u}{2r} B_{2r} B_{2u-2r}. \tag{54}$$

Now write

$$b_r = (-1)^{r-1} B_r \tag{55}$$

and replace in (52) to produce

$$(r+2)b_{r+1} = (r+1)b_r + \sum_{j=1}^{r} \binom{r+1}{j} b_j b_{r+1-j}. \tag{56}$$

This recurrence gives another proof of Corollary 6.

Now replace $r+1$ by n in (52) to obtain

$$(n+1)B_n = -nB_{n-1} - \sum_{j=1}^{n-1} \binom{n}{j} B_j B_{n-j} \tag{57}$$

that holds for $n \geq 1$. Write the sum as

$$\begin{aligned} \sum_{j=1}^{n-1} \binom{n}{j} B_j B_{n-j} &= \sum_{j=2}^{n-2} \binom{n}{j} B_j B_{n-j} + 2nB_1 B_{n-1} \\ &= \sum_{j=2}^{n-2} \binom{n}{j} B_j B_{n-j} - nB_{n-1}. \end{aligned} \tag{58}$$

Then (57) becomes

$$(n+1)B_n = -\sum_{j=2}^{n-2} \binom{n}{j} B_j B_{n-j}. \tag{59}$$

This relation is valid only for $n \geq 3$. *What happens for* $n = 1$ *and* $n = 2$*?*