THE BERNOULLI POLYNOMIALS

The **Bernoulli numbers** have been defined here by the exponential generating function

(1) $$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$$

The question considered here is how to introduce a *parameter* into these numbers. One option is to multiply the left-hand side of (1) by a simple function and to see what happens. Naturally, from the point of view of generating functions, the simplest possible function is an exponential. Therefore consider the expansion

(2) $$\frac{t}{e^t - 1} \times e^{xt} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

where the coefficients on the right-hand side depend on this parameter x.

The rule to multiply the series on the left is

(3) $$\left(\sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \right) \times \left(\sum_{j=0}^{\infty} b_j \frac{t^j}{j!} \right) = \sum_{r=0}^{\infty} \left[\sum_{i=0}^{r} \binom{r}{i} a_i b_{r-i} \right] \frac{t^r}{r!}$$

and applied to (2) gives

(4) $$\begin{aligned} \frac{t}{e^t - 1} \times e^{xt} &= \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \right) \times \left(\sum_{j=0}^{\infty} \frac{x^j t^j}{j!} \right) \\ &= \sum_{r=0}^{\infty} \left[\sum_{i=0}^{r} \binom{r}{i} B_i x^{r-i} \right] \frac{t^r}{r!}. \end{aligned}$$

It follows that

(5) $$B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$$

is a polynomial in x, called the **Bernoulli polynomial**.

An alternative approach to these polynomials is to start with the generating function (2), without any assumptions on the functions $B_k(x)$ and compute the first one by

(6) $$B_0(x) = \lim_{t \to 0} \frac{t e^{xt}}{e^t - 1} = 1$$

and then differentiate (2) with respect to x (after all the exponential function appears here, so this must be simple) to obtain

(7) $$\sum_{k=0}^{\infty} B_k'(x) \frac{t^k}{k!} = \frac{t^2 e^{xt}}{e^t - 1}.$$

The right-hand side is

$$t \times \frac{te^{xt}}{e^t - 1} = t \times \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} = \sum_{k=0}^{\infty} B_k(x) \frac{t^{k+1}}{k!} = \sum_{k=1}^{\infty} B_{k-1}(x) \frac{t^k}{(k-1)!}, \tag{8}$$

and using $B_0(x) = 1$ it is seen that the sum on the left-hand side starts at $k = 1$, leading to

$$\sum_{k=1}^{\infty} B_k'(x) \frac{t^k}{k!} = \sum_{k=1}^{\infty} B_{k-1}(x) \frac{t^k}{(k-1)!}. \tag{9}$$

Now match coefficients of equal powers of t to produce

$$\frac{B_k'(x)}{k!} = \frac{B_{k-1}(x)}{(k-1)!} \tag{10}$$

This is written as a theorem.

Theorem 1. *The functions $B_k(x)$ appearing in the expansion* (2) *satisfy the recurrence*

$$B_k'(x) = kB_{k-1}(x), \qquad \text{for } k \geq 1. \tag{11}$$

In particular, the initial condition $B_0(x) = 1$, shows that $B_k(x)$ is a polynomial of degree k. The expansion (2) *gives the special value*

$$B_k(0) = B_k \tag{12}$$

that can be used as a condition to determine the constant of integration in the process of determining $B_k(x)$ from $B_{k-1}(x)$ using (11).

The presence of the exponential factor e^{xt} suggests to compute the generating functions of $B_k(x+1)$. This is

$$\sum_{k=0}^{\infty} B_k(x+1) \frac{t^k}{k!} = \frac{te^{(x+1)t}}{e^t - 1} = \frac{t(e^t - 1 + 1)e^{xt}}{e^t - 1} = te^{xt} + \frac{te^{xt}}{e^t - 1}. \tag{13}$$

Now compare the coefficients of t^k to obtain the relation stated in the next theorem.

Theorem 2. *For $k \geq 0$, the Bernoulli polynomials satisfy the relation*

$$B_k(x+1) = B_k(x) + kx^{k-1}. \tag{14}$$

Now that we see the expression (14) it seems natural to sum over x, after all some parts will telescope. To make it consistent with previous notation, change (14) to

$$B_{a+1}(k+1) = B_{a+1}(k) + (a+1)k^a \tag{15}$$

(the pair $\{k, x\}$ was changed to $\{a+1, k\}$). Then, summing from $k = 0$ to n gives

$$B_{a+1}(n+1) - B_{a+1}(0) = (a+1)\sum_{k=0}^{n} k^a. \tag{16}$$

that implies

$$\sum_{k=1}^{n} k^a = \frac{B_{a+1}(n+1) - B_{a+1}}{a+1}. \tag{17}$$

This is the expression that we were looking for at the beginning of the semester.

Many other identities for the Bernoulli polynomials can be established using the generating function. The next example illustrates this point:

Theorem 3. *The Bernoulli polynomials satisfy the identity*

$$B_k(x) + B_k\left(x + \tfrac{1}{2}\right) = 2^{1-k} B_k(2x), \quad \text{for } k \geq 0. \tag{18}$$

The generating function of the left-hand side is

$$\begin{aligned}
\sum_{k=0}^{\infty} \left[B_k(x) + B_k\left(x + \tfrac{1}{2}\right)\right] \frac{t^k}{k!} &= \frac{te^{xt}}{e^t - 1} + \frac{te^{(x+1/2)t}}{e^t - 1} \\
&= \frac{te^{xt}}{e^t - 1}\left(1 + e^{t/2}\right) \\
&= \frac{te^{xt}}{(e^{t/2} - 1)(e^{t/2} + 1)}\left(1 + e^{t/2}\right) \\
&= \frac{te^{xt}}{(e^{t/2} - 1)} \\
&= 2\frac{(t/2)e^{2x(t/2)}}{(e^{t/2} - 1)} \\
&= 2\sum_{k=0}^{\infty} B_k(2x)\frac{(t/2)^k}{k!} \\
&= \sum_{k=0}^{\infty} 2^{1-k} B_k(2x)\frac{t^k}{k!}
\end{aligned}$$

and the result is obtained by matching powers of t.

The identity (5) shows that the value of the Bernoulli polynomial at $x = 0$ is the Bernoulli numbers, that is,

$$B_k(0) = B_k. \tag{19}$$

Replacing this in (18) gives

$$B_k + B_k\left(\tfrac{1}{2}\right) = 2^{1-k} B_k. \tag{20}$$

This proves the next identity.

Theorem 4. *The Bernoulli polynomials satisfy*

$$B_k\left(\tfrac{1}{2}\right) = \left(2^{1-k} - 1\right) B_k, \quad \text{for } k \geq 0. \tag{21}$$

With the value of $B_k\left(\frac{1}{2}\right)$ we can replace $x = \frac{1}{2}$ in (18) to obtain

$$B_k\left(\tfrac{1}{2}\right) + B_k(1) = 2^{1-k}B_k(1). \tag{22}$$

This gives

$$B_k\left(\tfrac{1}{2}\right) = \left(2^{1-k} - 1\right)B_k(1). \tag{23}$$

Comparing with (21) produces

$$\left(2^{1-k} - 1\right)B_k(1) = \left(2^{1-k} - 1\right)B_k. \tag{24}$$

For $k \neq 1$ the common factor does not vanish, this gives $B_k(1) = B_k$. On the other hand

$$B_k(1) = B_0x + B_1 \quad \text{at } x = 1 \tag{25}$$

that produces

$$B_k(1) = \tfrac{1}{2}. \tag{26}$$

This is summarized as follows

$$B_k(1) = \begin{cases} B_k & \text{for } k \neq 1 \\ -B_k & \text{for } k = 1, \end{cases} \tag{27}$$

that can be written in the simpler form

$$B_k(1) = (-1)^k B_k. \tag{28}$$

This is valid for $k = 1$ and k even. For k odd $k \neq 1$, the right-hand side of (28) vanishes and the same is true for the left-hand side from (27). Therefore (28) holds for all values of k.

Theorem 5. *For $k \geq 0$*

$$B_k(1) = (-1)^k B_k. \tag{29}$$

It is easy to check that the relation (29) extends to the Bernoulli polynomials. Indeed, starting with the generating function

$$\sum_{k=0}^{\infty} B_k(1-x)\frac{t^k}{k!} = \frac{te^{(1-x)t}}{e^t - 1} \tag{30}$$

and writing $s = -t$ it becomes

$$\begin{aligned} \frac{te^{(1-x)t}}{e^t - 1} &= -\frac{se^{-s(1-x)}}{e^{-s} - 1} \\ &= \frac{se^{xs}}{e^s - 1} \\ &= \sum_{k=0}^{\infty} B_k(x)\frac{s^k}{k!} \\ &= \sum_{k=0}^{\infty} B_k(x)\frac{(-1)^k t^k}{k!}. \end{aligned} \tag{31}$$

Now compare the coefficients of t^k to obtain the next result.

Theorem 6. *The Bernoulli polynomials satisfy*

$$B_k(1-x) = (-1)^k B_k(x), \quad \textit{for } k \geq 0. \tag{32}$$