PROOF OF FORMULA 3.197.5

$$\int_0^\infty x^{\lambda-1}(1+x)^{\nu}(1+ax)^{\mu}\,dx = B(\lambda,-\mu-\nu-\lambda)\;{}_2F_1\left[-\mu,\lambda;-\mu-\nu;1-a\right]$$

The change of variables $t = x/(1+x)$ yields

$$\int_0^\infty x^{\lambda-1}(1+x)^{\nu}(1+ax)^{\mu}\,dx = \int_0^1 t^{\lambda-1}(1-t)^{-\lambda-\nu-\mu-1}\left[1-(1-a)t\right]^{\mu}\,dt.$$

The result now comes from the integral representation of the hypergeometric function

$${}_2F_1\left[\alpha,\beta;\gamma;z\right] = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{-\alpha}\,dx.$$