PROOF OF FORMULA 3.249.8

$$\int_{-\infty}^{\infty} \left(1 + \frac{x^2}{n-1} \right)^{-n/2} \, dx = \frac{\pi(n-1)}{\Gamma(n/2)} \Gamma\left(\frac{n-1}{2} \right)$$

Consider the more general problem

$$f(a,b) = \int_{-\infty}^{\infty} \left(1 + ax^2\right)^{-b} \, dx$$

Let $y = \sqrt{a}x$ to obtain

$$f(a,b) = \frac{2}{\sqrt{a}} \int_0^{\infty} (1+y^2)^{-b} \, dy.$$

The change of variables $t = y^2$ now gives

$$f(a,b) = \frac{1}{\sqrt{a}} \int_0^{\infty} \frac{t^{-1/2} \, dt}{(1+t)^b}.$$

Using the integral representation

$$B(a,b) = \int_0^{\infty} \frac{t^{a-1} \, dt}{(1+t)^{a+b}},$$

we obtain

$$f(a,b) = \frac{1}{\sqrt{a}} B\left(\tfrac{1}{2}, b - \tfrac{1}{2}\right) = \sqrt{\frac{\pi}{a}} \frac{\Gamma(b-\frac{1}{2})}{\Gamma(b)}.$$

The stated formula corresponds to $a = \frac{1}{n-1}$ and $b = \frac{n}{2}$.

The more general formula is

$$\int_{-\infty}^{\infty} \left(1 + ax^2\right)^{-b} \, dx = \sqrt{\frac{\pi}{a}} \frac{\Gamma(b-\frac{1}{2})}{\Gamma(b)}.$$