PROOF OF FORMULA 3.251.4

$$\int_0^\infty \frac{x^{2m}\,dx}{(ax^2+c)^n} = \frac{(2m-1)!!\,(2n-2m-3)!!}{2(2n-2)!!\,a^m c^{n-m-1}}\,\frac{\pi}{\sqrt{ac}}$$

Let $x = \sqrt{c}t/\sqrt{a}$ so that $ax^2 = ct^2$ to obtain

$$\int_0^\infty \frac{x^{2m}\,dx}{(ax^2+c)^n} = \frac{1}{c^{n-m-1}a^m\sqrt{ac}} J(n,m),$$

where

$$J(n,m) = \int_0^\infty \frac{t^{2m}\,dt}{(1+t^2)^n}.$$

To evaluate this integral, let $u = t^2$ to obtain

$$J(n,m) = \frac{1}{2}\int_0^\infty \frac{u^{m-1/2}\,du}{(1+u)^n} = \frac{1}{2}B\left(m+\tfrac{1}{2}, n-m-\tfrac{1}{2}\right),$$

using the integral representation 8.380.3 in the table

$$B(a,b) = \int_0^\infty \frac{t^{a-1}\,dt}{(1+t)^{a+b}}.$$

Conclude that

$$J(n,m) = \frac{\Gamma(m+1/2)\,\Gamma(n-m-1/2)}{2\Gamma(n)}.$$

The stated form of the integral follows from the expression $2^n n! = (2n)!!$.