PROOF OF FORMULA 3.311.10

$$\int_0^\infty \frac{e^{-px} - e^{-qx}}{1 - e^{-(p+q)x}}\, dx = \frac{\pi}{p+q} \cot \frac{\pi p}{p+q}$$

Let $t = e^{-x}$ to obtain

$$\int_0^\infty \frac{e^{-px} - e^{-qx}}{1 - e^{-(p+q)x}}\, dx = \int_0^1 \frac{t^{p-1} - t^{q-1}}{1 - t^{p+q}}\, dt.$$

The change of variables $s = t^{p+q}$ yields

$$\int_0^1 \frac{t^{p-1} - t^{q-1}}{1 - t^{p+q}}\, dt = \frac{1}{p+q} \int_0^1 \frac{s^{p/(p+q)-1} - s^{q/(p+q)-1}}{1-s}\, ds.$$

Formula 3.231.5 shows that this is

$$\frac{1}{p+q} \left(\psi \left(\frac{q}{p+q} \right) - \psi \left(\frac{p}{p+q} \right) \right) = \frac{\pi}{p+q} \cot \left(\frac{\pi p}{p+q} \right),$$

using $\psi(z) - \psi(1-z) = \pi \cot(\pi z)$.