FORMULA 3.311.5

$$\int_0^\infty \frac{1-e^{\nu x}}{e^x-1}\,dx = \psi(\nu)+\gamma+\pi\cot(\pi\nu)$$

Let $t = e^{-x}$ to obtain

$$\int_0^\infty \frac{1-e^{\nu x}}{e^x-1}\,dx = \int_0^1 \frac{1-t^{-\nu}}{1-t}\,dt.$$

The integral representation

$$\psi(a) = -\int_0^1 \left(\frac{1}{\ln x}+\frac{x^{a-1}}{1-x}\right)\,dx,$$

is given as 8.361.4.

Then

$$\int_0^1 \frac{x^{\mu-1}-x^{\nu-1}}{1-x}\,dx \quad = \quad -\int_0^1 \left(\frac{1}{\ln x}+\frac{x^{\nu-1}}{1-x}\right)\,dx + \int_0^1 \left(\frac{1}{\ln x}+\frac{x^{\mu-1}}{1-x}\right)\,dx$$

gives

$$\int_0^1 \frac{x^{\mu-1}-x^{\nu-1}}{1-x}\,dx = \psi(\nu)-\psi(\mu).$$

Take $\mu = 1$ and replace ν by $1-\nu$ to get

$$\int_0^1 \frac{1-x^{-\nu}}{1-x}\,dx = \psi(1-\nu)-\psi(1).$$

The result follows from $\psi(1-\nu) = \psi(\nu)+\pi\cot(\pi\nu)$ and $\psi(1) = -\Gamma'(1) = -\gamma$.