PROOF OF FORMULA 3.312.2

$$\int_0^\infty \frac{(1-e^{-\alpha x})(1-e^{-\beta x})e^{-px}}{1-e^{-x}}\,dx = \psi(p+\alpha)+\psi(p+\beta)-\psi(p+\alpha+\beta)-\psi(p)$$

The change of variables $t = e^{-x}$ yields

$$\int_0^\infty \frac{(1-e^{-\alpha x})(1-e^{-\beta x})e^{-px}}{1-e^{-x}}\,dx = \int_0^1 \frac{t^{p-1}(1-t^\alpha-t^\beta+t^{\alpha+\beta})}{1-t}dt.$$

Write the integrand as

$$-\frac{(t^{p-1}-1)-(t^{p-1-\alpha}-1)-(t^{p-1+\beta}-1)+(t^{p-1+\alpha+\beta}-1)}{t-1},$$

and use the integral representation

$$\psi(z) = \int_0^1 \frac{t^{z-1}-1}{t-1}dt - \gamma$$

for the polygamma function to obtain the result.