PROOF OF FORMULA 3.324.1

$$\int_0^\infty \exp\left(-\frac{b}{4x} - ax\right)\, dx = \sqrt{\frac{b}{a}} K_1\left(\sqrt{ab}\right)$$

The *Bessel function of imaginary argument* is defined by the integral representation

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-z^2/4t}\, \frac{dt}{t^{\nu+1}}.$$

The change of variables $s = 1/t$ yields

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty s^{\nu-1}\, e^{-1/s-z^2 s/4}\, ds.$$

Let $s = 4ax/z^2$ to obtain

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \left(\frac{4a}{z^2}\right)^\nu \int_0^\infty x^{\nu-1}\, e^{-z^2/4ax-ax}\, dx.$$

Now take $\nu = 1$ and $z = \sqrt{ab}$ to obtain the result.