PROOF OF FORMULA 3.325

$$\int_0^\infty e^{-ax^2-b/x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}$$

Complete the square to obtain

$$\int_0^\infty e^{-ax^2-b/x^2}\,dx = e^{-2\sqrt{ab}}J,$$

where

$$J = \int_0^\infty e^{-(\sqrt{a}x-\sqrt{b}/x)^2}\,dx.$$

Now let $t = \frac{\sqrt{b}}{\sqrt{a}x}$ to obtain

$$J = \frac{\sqrt{b}}{\sqrt{a}}\int_0^\infty e^{-(\sqrt{a}t-\sqrt{b}/t)^2}\,\frac{dt}{t^2}.$$

Taking the average of these two forms for J,

$$J = \frac{1}{2}\int_0^\infty e^{-(\sqrt{a}x-\sqrt{b}/x)^2}\left(1+\frac{\sqrt{b}}{\sqrt{a}x^2}\right)\,dx.$$

The change of variables $u = \sqrt{a}x - \sqrt{b}/x$ gives

$$J = \frac{1}{2\sqrt{a}}\int_{-\infty}^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2\sqrt{a}}.$$