

PROOF OF FORMULA 3.364.3

$$\int_0^\infty \frac{e^{-px} dx}{\sqrt{x(x+a)}} = e^{ap/2} K_0\left(\frac{ap}{2}\right)$$

The change of variable $x = ta$ yields

$$\int_0^\infty \frac{e^{-px} dx}{\sqrt{x(x+a)}} = \int_0^\infty \frac{e^{-pat} dt}{\sqrt{t(1+t)}}.$$

Now let $\sigma = 2t$ and observe that $t^2 + t = \frac{1}{4}[(\sigma + 1)^2 - 1]$, to obtain

$$\int_0^\infty \frac{e^{-pat} dt}{\sqrt{t(1+t)}} = \int_0^\infty \frac{e^{-pa\sigma/2} d\sigma}{\sqrt{(\sigma + 1)^2 - 1}}.$$

The change of variable $u = \sigma + 1$ produces

$$\int_0^\infty \frac{e^{-pa\sigma/2} d\sigma}{\sqrt{(\sigma + 1)^2 - 1}} = e^{pa/2} \int_1^\infty \frac{e^{-pau/2} du}{\sqrt{u^2 - 1}}.$$

The integral representation

$$K_\nu(z) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zt} (t^2 - 1)^{\nu - \frac{1}{2}} dt,$$

that, for $\nu = 0$ becomes

$$K_0(z) = \int_1^\infty \frac{e^{-zt} dt}{\sqrt{t^2 - 1}}$$

gives the result.