

PROOF OF FORMULA 3.411.14

$$\int_0^{\infty} \frac{x^2 e^{-nx}}{1 - e^{-x}} dx = 2 \sum_{k=n}^{\infty} \frac{1}{k^3} = 2 \left(\zeta(3) - \sum_{k=1}^{n-1} \frac{1}{k^3} \right)$$

Formula 3.411.7 states that

$$\int_0^{\infty} \frac{x^{\nu-1} e^{-\mu x}}{1 - e^{-bx}} dx = \frac{\Gamma(\nu)}{b^{\nu}} \zeta\left(\nu, \frac{\mu}{b}\right).$$

The special case $\nu = 3$, $\mu = n$ and $b = 1$ give

$$\int_0^{\infty} \frac{x^2 e^{-nx}}{1 - e^{-x}} dx = \Gamma(3) \zeta(3, n).$$

The answer comes from $\Gamma(3) = 2$ and

$$\zeta(3, n) = \sum_{k=0}^{\infty} \frac{1}{(k+n)^3} = \sum_{k=n}^{\infty} \frac{1}{k^3}.$$