

PROOF OF FORMULA 3.411.15

$$\int_0^\infty \frac{x^2 e^{-nx}}{1 + e^{-x}} dx = 2 \sum_{k=n}^\infty \frac{(-1)^{n+k}}{k^3} = (-1)^{n+1} \left(\frac{3}{2} \zeta(3) + 2 \sum_{k=1}^{n-1} \frac{(-1)^k}{k^3} \right)$$

Write

$$\int_0^\infty \frac{x^2 e^{-nx}}{1 + e^{-x}} dx = \int_0^\infty \frac{x^2 e^{-(n-1)x}}{1 + e^x} dx,$$

and use 3.411.8

$$\int_0^\infty \frac{x^{m-1} e^{-px}}{1 + e^x} dx = (m-1)! \sum_{k=1}^\infty \frac{(-1)^{k-1}}{(k+p)^m}$$

to obtain

$$\int_0^\infty \frac{x^2 e^{-nx}}{1 + e^{-x}} dx = 2 \sum_{k=1}^\infty \frac{(-1)^{k-1}}{(k+n-1)^3} = 2 \sum_{k=n}^\infty \frac{(-1)^{k+n}}{k^3}.$$

The result is simplified using

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^3} = -\frac{3}{4} \sum_{k=1}^\infty \frac{1}{k^3},$$

that is obtained by the usual even-odd splitting trick.