PROOF OF FORMULA 3.411.17

$$\int_0^\infty \frac{x^3 e^{-nx}\, dx}{1-e^{-x}} = \frac{\pi^4}{15} - 6\sum_{k=1}^{n-1} \frac{1}{k^4}$$

Formula 3.411.6 states that

$$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{1-be^{-x}}\, dx = \Gamma(\nu)\sum_{k=0}^{\infty} \frac{b^k}{(\mu+k)^\nu}.$$

The special case $\nu = 4$, $\mu = n$ and $b = 1$ yields

$$\int_0^\infty \frac{x^3 e^{-nx}}{1-e^{-x}}\, dx = \Gamma(4)\sum_{k=0}^{\infty} \frac{1}{(\mu+k)^4}.$$

The value $\Gamma(4) = 6$ and

$$\zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90},$$

give the result.