PROOF OF FORMULA 3.411.18

$$\int_0^\infty \frac{x^3 e^{-nx}\,dx}{1+e^{-x}} = 6\sum_{k=n}^\infty \frac{(-1)^{n+k}}{k^4}$$

Write

$$\int_0^\infty \frac{x^3 e^{-nx}\,dx}{1+e^{-x}} = \int_0^\infty \frac{x^3 e^{-(n-1)x}\,dx}{1+e^{x}}.$$

Formula 3.411.8 states that

$$\int_0^\infty \frac{x^{n-1} e^{-px}\,dx}{1+e^{x}} = (n-1)!\sum_{k=1}^\infty \frac{(-1)^{k-1}}{(k+p)^n}.$$

Replacing $n=4$ and $p=n-1$ gives

$$\int_0^\infty \frac{x^3 e^{-(n-1)x}\,dx}{1+e^{x}} = 6\sum_{k=1}^\infty \frac{(-1)^{k-1}}{(k+n-1)^4}.$$

Now shift the index to obtain the result.