

PROOF OF FORMULA 3.411.19

$$\int_0^{\infty} e^{-px} (e^{-x} - 1)^n \frac{dx}{x} = - \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(p + n - k)$$

Define

$$I(p) := \int_0^{\infty} e^{-px} (e^{-x} - 1)^n \frac{dx}{x},$$

and differentiate with respect to p to obtain

$$\begin{aligned} I'(p) &= - \int_0^{\infty} e^{-px} (e^{-x} - 1)^n dx \\ &= - \sum_{k=0}^n (-1)^k \binom{n}{k} \int_0^{\infty} e^{-(p+n-k)x} dx \\ &= - \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{p+n-k}. \end{aligned}$$

Therefore

$$I(p) = - \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(p + n - k) + C.$$

The constant of integration C is determined from the condition $I(+\infty) = 0$. Now observe that

$$\prod_{k=0}^n (p + n - k)^{(-1)^k \binom{n}{k}} \sim p^{\sum_{k=0}^n (-1)^k \binom{n}{k}} = 1.$$

Therefore, $C = 0$.