

PROOF OF FORMULA 3.411.21

$$\int_0^{\infty} x^{n-1} \frac{1 - e^{-mx}}{1 - e^{-x}} dx = -(n-1)! \sum_{k=1}^m \frac{1}{k^n}$$

Write the integral as

$$\int_0^{\infty} x^{n-1} \frac{1 - e^{-mx}}{1 - e^{-x}} dx = - \int_0^{\infty} x^{n-1} e^{-mx} \frac{e^{mx} - 1}{e^x - 1} dx.$$

The rational function part of the integrand can be expanded to obtain

$$\int_0^{\infty} x^{n-1} \frac{1 - e^{-mx}}{1 - e^{-x}} dx = - \sum_{k=0}^{m-1} \int_0^{\infty} x^{n-1} e^{-(m-k)x} dx.$$

The change of variables $t = (m - k)x$ gives

$$\int_0^{\infty} x^{n-1} \frac{1 - e^{-mx}}{1 - e^{-x}} dx = - \sum_{k=0}^{m-1} \frac{1}{(m-k)^n} \int_0^{\infty} t^{n-1} e^{-t} dt.$$

The integral is recognized as $\Gamma(n) = (n-1)!$, to conclude.