

PROOF OF FORMULA 3.411.23

$$\int_{-\infty}^{\infty} \frac{x e^{\mu x} dx}{b + e^x} = \frac{\pi b^{\mu-1}}{\sin \pi \mu} [\ln b - \pi \cot \pi \mu]$$

The change of variables $x = t + \ln b$ yields $e^x = b e^t$ and

$$\int_{-\infty}^{\infty} \frac{x e^{\mu x} dx}{b + e^x} = \ln b \int_{-\infty}^{\infty} \frac{e^{\mu t} dt}{1 + e^t} + \int_{-\infty}^{\infty} \frac{t e^{\mu t} dt}{1 + e^t}.$$

The change of variables $s = e^t$ shows that the first integral is

$$\int_{-\infty}^{\infty} \frac{e^{\mu t} dt}{1 + e^t} = \int_0^{\infty} \frac{s^{\mu-1} ds}{1 + s} = B(\mu, 1 - \mu) = \frac{\pi}{\sin \pi \mu}.$$

The value of the second integral comes from differentiating the first one with respect to μ :

$$\int_{-\infty}^{\infty} \frac{t e^{\mu t} dt}{1 + e^{-t}} = -\frac{\pi^2 \cot \pi \mu}{\sin \pi \mu}.$$

Replacing, produces the result.