

PROOF OF FORMULA 3.411.26

$$\int_0^{\infty} x e^{-x} \frac{1 - e^{-x}}{1 + e^{-3x}} dx = \frac{2\pi^2}{27}$$

Define

$$J_n := \int_0^{\infty} \frac{x e^{-nx} dx}{1 + e^{-3x}},$$

so that the requested evaluation is $J_1 - J_2$.

The change of variables $t = 3x$ and expanding the integrand in series gives

$$J_1 = \frac{1}{9} \sum_{k=0}^{\infty} (-1)^k \int_0^{\infty} t e^{-(k+1/3)t} dt.$$

The change of variables $u = (k + 1/3)t$ gives

$$J_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k + 1)^2}.$$

The same type of argument produces

$$J_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k + 2)^2}.$$

It follows that

$$\begin{aligned} \int_0^{\infty} x e^{-x} \frac{1 - e^{-x}}{1 + e^{-3x}} dx = J_1 - J_2 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k + 1)^2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k + 2)^2} \\ &= - \sum_{k=0}^{\infty} \frac{(-1)^{3k+1}}{(3k + 1)^2} - \sum_{k=0}^{\infty} \frac{(-1)^{3k+2}}{(3k + 2)^2} \\ &= - \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} + \sum_{k=1}^{\infty} \frac{(-1)^{3k}}{(3k)^2} \\ &= - \frac{8}{9} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}. \end{aligned}$$

The value

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}$$

gives the result.