PROOF OF FORMULA 3.427.1

$$\int_0^\infty \left[\frac{e^{-x}}{x} + \frac{e^{-ax}}{e^{-x}-1} \right] dx = \psi(a)$$

Start with the representation

$$\psi(a) = \int_0^\infty \left[e^{-x} - (1+x)^{-a} \right] \frac{dx}{x},$$

given in entry 3.429. Write this as

$$\psi(a) = \lim_{\delta\to 0} \int_\delta^\infty \frac{e^{-x}}{x}\, dx - \lim_{\delta\to 0} \int_\delta^\infty \frac{dx}{x(1+x)^a}.$$

Let $t = 1+x$ in the second integral to obtain

$$\psi(a) = \lim_{\delta\to 0} \left[\int_{\ln(1+\delta)}^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-ax}}{1-e^{-x}} \right) dx + \int_\delta^{\ln(1+\delta)} \frac{e^{-x}}{x}\, dx \right]$$

Conclude with the estimate

$$\left| \int_\delta^{\ln(1+\delta)} \frac{e^{-x}}{x} \right| dx \leq \int_{\ln(1+\delta)}^\delta \frac{dx}{x} = \ln\left(\frac{\delta}{\ln(1+\delta)} \right),$$

that converges to 0 as $\delta \to 0$.