PROOF OF FORMULA 3.442.3

$$\int_0^\infty \left(e^{-px} - \frac{1}{1+a^2x^2} \right) \frac{dx}{x} = -\gamma + \ln \frac{a}{p}$$

The change of variables $t = px$ yields

$$\int_0^\infty \left(e^{-px} - \frac{1}{1+a^2x^2} \right) \frac{dx}{x} = \int_0^\infty \left(e^{-t} - \frac{1}{1+c^2t^2} \right) \frac{dt}{t}$$

with $c = a/p$. This can be written as

$$\int_0^\infty \left(e^{-t} - \frac{1}{1+c^2t^2} \right) \frac{dt}{t} = \int_0^\infty \left(e^{-t} - \frac{1}{1+t} \right) \frac{dt}{t} + \int_0^\infty \left(\frac{1}{1+t} - \frac{1}{1+c^2t^2} \right) \frac{dt}{t}.$$

Formula 3.435.3 states that the first integral is $-\gamma$ and the partial fraction decomposition

$$\left(\frac{1}{1+t} - \frac{1}{1+c^2t^2} \right) \frac{1}{t} = -\frac{1}{1+t} + \frac{c^2t}{1+c^2t^2},$$

shows that

$$\int_0^b \left(\frac{1}{1+t} - \frac{1}{1+c^2t^2} \right) \frac{dt}{t} = \frac{1}{2} \ln \frac{1+c^2b^2}{(1+b)^2}.$$

Passing to the limit as $b \to \infty$ shows that the second integral is $\ln c$. This gives the result.