

**PROOF OF FORMULA 3.451.1**

$$\int_0^{\infty} x e^{-x} \sqrt{1 - e^{-x}} dx = \frac{4}{9}(4 - 3 \ln 2)$$

Observe that

$$\int_0^{\infty} x e^{-x} \sqrt{1 - e^{-x}} dx = -h'(1),$$

where

$$h(a) = \int_0^{\infty} e^{-ax} \sqrt{1 - e^{-x}} dx.$$

The change of variables  $t = e^{-x}$  gives

$$h(a) = \int_0^1 t^{a-1} (1-t)^{1/2} dt = B(a, \frac{3}{2}).$$

Differentiation yields

$$h'(a) = h(a) [\psi(a) - \psi(a + \frac{3}{2})],$$

where  $\psi = \Gamma'/\Gamma$  is the polygamma function. Therefore

$$\int_0^{\infty} x e^{-x} \sqrt{1 - e^{-x}} dx = -\frac{\Gamma(1)\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})} [\psi(1) - \psi(\frac{5}{2})].$$

The value  $\psi(1) = -\gamma$  and  $\psi(\frac{5}{2}) = -\gamma - 2 \ln 2 + \frac{8}{3}$  give the result. To obtain the expression for  $\psi(\frac{5}{2})$ , differentiate the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2}),$$

to produce

$$2\psi(2z) = 2 \ln 2 + \psi(z) + \psi(z + \frac{1}{2}).$$

The simplification also makes use of the relation

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}.$$