

PROOF OF FORMULA 3.452.2

$$\int_0^{\infty} \frac{x^2 dx}{\sqrt{e^x - 1}} = \frac{\pi}{3} (12 \ln^2 2 + \pi^2)$$

The integral is

$$\int_0^{\infty} \frac{x^2 dx}{\sqrt{e^x - 1}} = \int_0^{\infty} \frac{x^2 e^{-x/2} dx}{\sqrt{1 - e^{-x}}},$$

and this can be expressed as $h''(\frac{1}{2})$, where

$$h(a) = \int_0^{\infty} \frac{e^{-ax} dx}{\sqrt{1 - e^{-x}}}.$$

To evaluate $h(a)$ let $t = e^{-x}$ to obtain

$$h(a) = \int_0^1 t^{a-1} (1-t)^{-1/2} dt = B(a, \frac{1}{2}) = \frac{\Gamma(a)\sqrt{\pi}}{\Gamma(a + \frac{1}{2})}.$$

Logarithmic differentiation yields

$$h'(a) = h(a) [\psi(a) - \psi(a + \frac{1}{2})]$$

and

$$h''(a) = h'(a) [\psi(a) - \psi(a + \frac{1}{2})] + h(a) [\psi'(a) - \psi'(a + \frac{1}{2})].$$

The values $\psi(1) = -\gamma$, $\psi(\frac{1}{2}) = -\gamma - 2 \ln 2$, $\psi'(\frac{1}{2}) = \pi^2/2$ and $\psi'(1) = \pi^2/6$ yield the result.