

PROOF OF FORMULA 3.452.3

$$\int_0^{\infty} \frac{x e^{-x} dx}{\sqrt{e^x - 1}} = \frac{\pi}{2} (2 \ln 2 - 1)$$

Write the problem as

$$\int_0^{\infty} \frac{x e^{-x} dx}{\sqrt{e^x - 1}} = \int_0^{\infty} \frac{x e^{-3x/2} dx}{\sqrt{1 - e^{-x}}}.$$

Now define

$$f(a) := \int_0^{\infty} \frac{e^{-ax} dx}{\sqrt{1 - e^{-x}}},$$

so that

$$f'(a) = - \int_0^{\infty} \frac{x e^{-ax} dx}{\sqrt{1 - e^{-x}}}.$$

Then

$$\int_0^{\infty} \frac{x e^{-x} dx}{\sqrt{e^x - 1}} = -f'(\frac{3}{2}).$$

Now let $u = e^{-x}$ to obtain

$$f(a) = \int_0^1 u^{a-1} (1-u)^{-1/2} du = B(a, \frac{1}{2}) = \frac{\Gamma(a) \Gamma(\frac{1}{2})}{\Gamma(a + \frac{1}{2})}.$$

The integral representation for the beta function

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$$

was used. Logarithmic differentiation gives

$$f'(a) = f(a) [\psi(a) - \psi(a + \frac{1}{2})].$$

The values $f(\frac{3}{2}) = \frac{\pi}{2}$ and

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \text{ and } \psi(n + \frac{1}{2}) = -\gamma + 2 \left(\sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right),$$

give

$$\psi(\frac{3}{2}) = 2 - \gamma - 2 \ln 2 \text{ and } \psi(2) = 1 - \gamma.$$

Therefore

$$f'(\frac{3}{2}) = \frac{\pi}{2} ((2 - \gamma - 2 \log 2) - (1 - \gamma)) = \frac{\pi}{2} (1 - 2 \log 2).$$

This completes the evaluation.