

**PROOF OF FORMULA 3.452.4**

$$\int_0^\infty \frac{x e^{-x} dx}{\sqrt{e^{2x} - 1}} = 1 - \ln 2$$

The change of variables  $t = 2x$  gives

$$\int_0^\infty \frac{x e^{-x} dx}{\sqrt{e^x - 1}} = \frac{1}{4} \int_0^\infty \frac{t e^{-t} dt}{\sqrt{1 - e^{-t}}}.$$

Now define

$$f(a) := \int_0^\infty \frac{e^{-ax} dx}{\sqrt{1 - e^{-x}}},$$

so that

$$f'(a) = - \int_0^\infty \frac{x e^{-ax} dx}{\sqrt{1 - e^{-x}}}.$$

Then

$$\int_0^\infty \frac{x e^{-x} dx}{\sqrt{e^{2x} - 1}} = -\frac{1}{4} f'(1).$$

Now let  $u = e^{-x}$  to obtain

$$f(a) = \int_0^1 u^{a-1} (1-u)^{-1/2} du = B(a, \frac{1}{2}) = \frac{\Gamma(a) \Gamma(\frac{1}{2})}{\Gamma(a + \frac{1}{2})}.$$

The integral representation for the beta function

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$$

was used. Logarithmic differentiation gives

$$f'(a) = f(a) [\psi(a) - \psi(a + \frac{1}{2})].$$

The values  $f(1) = 2$  and

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \text{ and } \psi(n + \frac{1}{2}) = -\gamma + 2 \left( \sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right),$$

give

$$\psi(1) = -\gamma \text{ and } \psi(\frac{3}{2}) = 2 - \gamma - 2 \ln 2.$$

Therefore

$$f'(1) = 2(-\gamma - (2 - \gamma - 2 \ln 2)) = -4(1 - \ln 2).$$

This completes the evaluation.