

**PROOF OF FORMULA 3.452.5**

$$\int_0^\infty \frac{x e^{-2x} dx}{\sqrt{e^x - 1}} = \frac{\pi}{16} (12 \ln 2 - 7)$$

Write the problem as

$$\int_0^\infty \frac{x e^{-2x} dx}{\sqrt{e^x - 1}} = \int_0^\infty \frac{x e^{-5x/2} dx}{\sqrt{1 - e^{-x}}}.$$

Now define

$$f(a) := \int_0^\infty \frac{e^{-ax} dx}{\sqrt{1 - e^{-x}}},$$

so that

$$f'(a) = - \int_0^\infty \frac{x e^{-ax} dx}{\sqrt{1 - e^{-x}}}.$$

Then

$$\int_0^\infty \frac{x e^{-x} dx}{\sqrt{e^x - 1}} = -f'(\frac{5}{2}).$$

Now let  $u = e^{-x}$  to obtain

$$f(a) = \int_0^1 u^{a-1} (1-u)^{-1/2} du = B(a, \frac{1}{2}) = \frac{\Gamma(a) \Gamma(\frac{1}{2})}{\Gamma(a + \frac{1}{2})}.$$

The integral representation for the beta function

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$$

was used. Logarithmic differentiation gives

$$f'(a) = f(a) [\psi(a) - \psi(a + \frac{1}{2})].$$

The values  $f(\frac{5}{2}) = \frac{3\pi}{8}$  and

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \text{ and } \psi(n + \frac{1}{2}) = -\gamma + 2 \left( \sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right),$$

give

$$\psi(\frac{5}{2}) = \frac{8}{3} - \gamma - 2 \ln 2 \text{ and } \psi(3) = \frac{3}{2} - \gamma.$$

Therefore

$$f'(\frac{3}{2}) = \frac{3\pi}{8} \left( (\frac{8}{3} - \gamma - 2 \log 2) - (\frac{3}{2} - \gamma) \right) = -\frac{3\pi}{4} (\frac{7}{12} - \ln 2).$$

This completes the evaluation.