PROOF OF FORMULA 3.511.10

$$\int_0^\infty \frac{\sinh ax \, \sinh bx}{\cosh^2 bx} \, dx = \frac{\pi a}{2b^2} \sec \frac{\pi a}{2b}$$

The change of variables $u = bx$ shows that this entry is equivalent to

$$\int_0^\infty \frac{\sinh cu \, \sinh u}{\cosh^2 u} \, du = \frac{\pi c}{2} \sec \frac{\pi c}{2}$$

where $c = a/b$. To prove this formula write it as

$$\int_0^\infty \frac{\sinh cu \, \sinh u}{\cosh^2 u} \, du = \int_0^\infty \frac{[e^{cu} - e^{-cu}] \, [e^u - e^{-u}]}{[e^u + e^{-u}]^2}.$$

The change of variable $t = e^{-u}$ gives

$$\int_0^\infty \frac{\sinh cu \, \sinh u}{\cosh^2 u} \, du = \int_0^1 \frac{t^{-c}(1 - t^{2c})(1 - t^2)}{(1 + t^2)^2} \, dt.$$

Now let $s = t^2$ and use the representation

$$B(x, y) = \int_0^1 \frac{t^{x-1} + t^{y-1}}{(1 + t)^{x+y}} \, dt$$

to obtain

$$\int_0^\infty \frac{\sinh cu \, \sinh u}{\cosh^2 u} \, du = \frac{1}{2} \left[B\left(\frac{c+3}{2}, \frac{1-c}{2} \right) - B\left(\frac{c+1}{2}, \frac{3-c}{2} \right) \right].$$

The result now follows by writing the beta function in terms of the gamma function and using

$$\Gamma(\tfrac{1}{2} + x)\Gamma(\tfrac{1}{2} - x) = \pi \sec \pi x.$$