PROOF OF FORMULA 3.511.9

$$\int_{-\infty}^{\infty} \frac{\sinh^2 ax}{\sinh^2 x}\, dx = 1 - \pi a \cot \pi a$$

Write the integral as

$$\int_{-\infty}^{\infty} \frac{\sinh^2 ax}{\sinh^2 x}\, dx = 2\int_0^{\infty} \frac{(e^{ax}-e^{-ax})^2}{(e^x - e^{-x})^2}\, dx.$$

The change of variable $t = e^{-x}$ followed by $s = t^2$ yields

$$\int_{-\infty}^{\infty} \frac{\sinh^2 ax}{\sinh^2 x}\, dx = \int_0^1 s^{-a}(1-s^a)^2(1-s)^{-2}\, ds.$$

The last integral is written as

$$\lim_{u \to -1} \int_0^1 (1-s)^{u-1}\left(s^{-a} - 2 + s^a\right) = \lim_{u \to -1} B(u, 1-a) - 2B(u,1) + B(u, 1+a).$$

This is

$$\int_{-\infty}^{\infty} \frac{\sinh^2 ax}{\sinh^2 x}\, dx = \lim_{u \to -1} \Gamma(u)\left[\frac{\Gamma(1-a)}{\Gamma(1-a+u)} + \frac{\Gamma(1+a)}{\Gamma(1+a+u)}\right] - \frac{2}{u}$$

and the last term is

$$\frac{\Gamma(2+u)}{u\Gamma(1-a+u)\Gamma(1+a+u)}\left[\frac{\Gamma(1-a)\Gamma(1+a+u)+\Gamma(1-a+u)\Gamma(1+a)}{1+u}\right]$$

and passing to the limit yields

$$\int_{-\infty}^{\infty} \frac{\sinh^2 ax}{\sinh^2 x}\, dx = 2 + a\left[\psi(a) - \psi(-a)\right].$$

The result follows by using the relations

$$\psi(x+1) = \psi(x) + \frac{1}{x} \text{ and } \psi(1-x) = \psi(x) + \pi \cot \pi x$$

that gives

$$\psi(-a) = \frac{1}{a} + \psi(a) + \pi \cot \pi a.$$