FORMULA 3.523.1

$$\int_0^\infty \frac{x^{b-1}\,dx}{\sinh ax} = \frac{2^b-1}{2^{b-1}a^b}\Gamma(b)\zeta(b)$$

The integral is

$$\int_0^\infty \frac{x^{b-1}\,dx}{\sinh ax} = 2\int_0^\infty \frac{x^{b-1}e^{-ax}}{1-e^{-2ax}}\,dx.$$

Expanding in a geometric series

$$\int_0^\infty \frac{x^{b-1}\,dx}{\sinh ax} = 2\sum_{k=0}^\infty \int_0^\infty x^{b-1}e^{-(2k+1)ax}\,dx.$$

The change of variables $u=(2k+1)ax$ gives

$$\int_0^\infty \frac{x^{b-1}\,dx}{\sinh ax} = \frac{2\Gamma(b)}{a^b}\sum_{k=0}^\infty \frac{1}{(2k+1)^b}.$$

To conclude, observe that

$$\sum_{k=0}^\infty \frac{1}{(2k+1)^b} + \sum_{k=1}^\infty \frac{1}{(2k)^b} = \zeta(b).$$

Therefore,

$$\sum_{k=0}^\infty \frac{1}{(2k+1)^b} = \frac{2^b-1}{2^b}\zeta(b).$$