

PROOF OF FORMULA 3.524.1

$$\int_0^{\infty} x^{s-1} \frac{\sinh bx}{\sinh cx} dx = \frac{\Gamma(s)}{(2c)^s} \left[\zeta \left(s, \frac{1}{2} \left(1 - \frac{b}{c} \right) \right) - \zeta \left(s, \frac{1}{2} \left(1 + \frac{b}{c} \right) \right) \right]$$

Write the integral as

$$\int_0^{\infty} x^{s-1} \frac{\sinh bx}{\sinh cx} dx = \int_0^{\infty} x^{s-1} (e^{bx} - e^{-bx}) e^{-cx} \frac{dx}{1 - e^{-2cx}}.$$

Now expand part of the integrand as a geometric series to produce

$$\int_0^{\infty} x^{s-1} \frac{\sinh bx}{\sinh cx} dx = \sum_{k=0}^{\infty} \int_0^{\infty} x^{s-1} e^{(b-c-2ck)x} dx - \sum_{k=0}^{\infty} \int_0^{\infty} x^{s-1} e^{(-b-c-2ck)x} dx.$$

Scaling the exponent yields

$$\int_0^{\infty} x^{s-1} \frac{\sinh bx}{\sinh cx} dx = \frac{\Gamma(s)}{(2c)^s} \left[\sum_{k=0}^{\infty} \frac{1}{\left(k + \frac{1}{2} \left(1 - \frac{b}{c} \right) \right)^s} - \sum_{k=0}^{\infty} \frac{1}{\left(k + \frac{1}{2} \left(1 + \frac{b}{c} \right) \right)^s} \right].$$

This is the stated expression, using the Hurwitz zeta function

$$\zeta(z, q) = \sum_{k=0}^{\infty} \frac{1}{(k+q)^z}.$$