PROOF OF FORMULA 3.527.1

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{\sinh^2(ax)} = \frac{4\Gamma(\mu)\zeta(\mu-1)}{(2a)^\mu}$$

The change of variable $t = ax$ shows that it is sufficient to consider the case $a = 1$. Start with

$$\int_0^\infty \frac{t^{\mu-1}\,dx}{\sinh^2 t} = 4\int_0^\infty \frac{t^{\mu-1}\,dx}{(e^t - e^{-t})^2},$$

and write the last integral as

$$J := \int_0^\infty \frac{t^{\mu-1}\,dt}{(e^t - e^{-t})^2} = \int_0^\infty \frac{t^{\mu-1}e^{-2t}\,dt}{(1 - e^{-2t})^2}.$$

Expand in a power series to obtain

$$J = \sum_{n=1}^\infty n \int_0^\infty t^{\mu-1}e^{-2nt}\,dt.$$

The change of variable $v = 2nt$ yields

$$J = \sum_{n=1}^\infty \frac{1}{n^{\mu-1}} \times \frac{1}{2^\mu} \int_0^\infty v^{\mu-1}e^{-v}\,dv.$$

The series gives the term $\zeta(\mu-1)$ and the integral is $\Gamma(\mu)$.