FORMULA 3.527.14

$$\int_0^\infty x^2 \, \frac{\sinh ax}{\cosh^2 ax} \, dx = \frac{4G}{a^3}$$

The change of variables $t = ax$ shows that it suffices to consider $a = 1$. Write the integral as

$$\int_0^\infty x^2 \, \frac{\sinh x}{\cosh^2 x} \, dx = 2 \int_0^\infty \frac{x^2(e^x - e^{-x})e^{-2x}}{(1+e^{-2x})^2} \, dx$$

and expand in a geometric series to obtain

$$\int_0^\infty x^2 \, \frac{\sinh x}{\cosh^2 x} \, dx = -2 \sum_{k=1}^\infty (-1)^k k \int_0^\infty x^2 (e^x - e^{-x}) e^{-2kx} \, dx.$$

Integrate term by term to obtain

$$\int_0^\infty x^2 \, \frac{\sinh x}{\cosh^2 x} \, dx = -4 \sum_{k=1}^\infty (-1)^k k \left[\frac{1}{(2k-1)^3} - \frac{1}{(2k+1)^3} \right].$$

To evaluate the series observe that

$$\begin{aligned} -4 \sum_{k=1}^\infty (-1)^k k \left[\frac{1}{(2k-1)^3} - \frac{1}{(2k+1)^3} \right] &= -4 \sum_{k=1}^\infty \frac{(-1)^k k}{(2k-1)^3} + 4 \sum_{k=1}^\infty \frac{(-1)^k k}{(2k+1)^3} \\ &= -4 \sum_{k=1}^\infty \frac{(-1)^k k}{(2k-1)^3} + 4 \sum_{k=2}^\infty \frac{(-1)^{k-1}(k-1)}{(2k-1)^3} \\ &= 4 \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2}. \end{aligned}$$

This proves the result.