

PROOF OF FORMULA 3.527.3

$$\int_0^\infty \frac{x^{\mu-1} dx}{\cosh^2 ax} = \frac{4}{(2a)^\mu} (1 - 2^{2-\mu}) \Gamma(\mu) \zeta(\mu - 1)$$

The scaling $t = ax$ shows that it suffices to assume $a = 1$. Write the integral as

$$\int_0^\infty \frac{x^{\mu-1} dx}{\cosh^2 x} = 4 \int_0^\infty \frac{x^{\mu-1} e^{-2x}}{(1 + e^{-2x})^2} dx.$$

Expand the integrand using

$$\frac{1}{(1 + u)^2} = \sum_{k=0}^\infty (-1)^k (k + 1) u^k$$

to produce

$$\int_0^\infty \frac{x^{\mu-1} dx}{\cosh^2 x} = 4 \sum_{k=0}^\infty (-1)^k (k + 1) \int_0^\infty x^{\mu-1} e^{-2(k+1)x} dx.$$

The change of variables $v = 2(k + 1)x$ yields

$$\int_0^\infty \frac{x^{\mu-1} dx}{\cosh^2 x} = -2^{2-\mu} \Gamma(\mu) \sum_{k=1}^\infty \frac{(-1)^k}{k^\mu}.$$

The last series can be expressed in terms of the Riemann zeta function by splitting the cases k even and odd to produce the identity

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^\mu} = (2^{1-\mu} - 1) \zeta(\mu - 1).$$

This gives the result.