PROOF OF FORMULA 3.527.4

$$\int_0^\infty \frac{x\,dx}{\cosh^2(ax)} = \frac{\ln 2}{a^2}$$

The change of variables $t = ax$ shows that it sufficient to ocnsider $a = 1$. Start with

$$\int_0^\infty \frac{t\,dt}{\cosh^2 t} = \int_0^\infty \frac{4te^{-2t}}{(1+e^{-2t})^2}\,dt$$

and expand the integrand in series to obtain

$$\int_0^\infty \frac{t\,dt}{\cosh^2 t} = 4\sum_{k=0}^\infty (-1)^k (k+1) \int_0^\infty te^{-2(k+1)t}\,dt.$$

The change of variables $v = 2(k+1)t$ yields

$$\int_0^\infty \frac{t\,dt}{\cosh^2 t} = \sum_{k=0}^\infty \frac{(-1)^k}{k+1} \int_0^\infty ve^{-v}\,dv.$$

The integral is $\Gamma(2) = 1$ and the series sums to $\ln 2$.