

PROOF OF FORMULA 3.541.10

$$\int_0^{\infty} e^{-qx} \frac{\sinh px}{\sinh qx} dx = \frac{1}{p} - \frac{\pi}{2q} \cot\left(\frac{\pi p}{2q}\right)$$

The change of variables $t = qx$ and the notation $c = p/q$ show that the evaluation is equivalent to

$$\int_0^{\infty} e^{-t} \frac{\sinh ct}{\sinh t} dt = \frac{1}{c} - \frac{\pi}{2} \cot\left(\frac{\pi c}{2}\right).$$

In order to establish this, write the integral as

$$\int_0^{\infty} e^{-t} \frac{\sinh ct}{\sinh t} dt = \int_0^{\infty} \frac{e^{-2t} (e^{ct} - e^{-ct})}{1 - e^{-2t}} dt.$$

Entry 3.311.7 states that

$$\int_0^{\infty} \frac{e^{-\mu x} - e^{-\nu x}}{1 - e^{-x}} dx = \psi(\nu) - \psi(\mu).$$

Therefore

$$\int_0^{\infty} e^{-t} \frac{\sinh ct}{\sinh t} dt = \frac{1}{2} \left[-\psi\left(1 - \frac{c}{2}\right) + \psi\left(1 + \frac{c}{2}\right) \right].$$

The result now follows by using $\psi(x+1) = \psi(x) + 1/x$ and $\psi(x) - \psi(1-x) = -\pi \cot \pi x$.