PROOF OF FORMULA 3.551.10

$$\int_0^\infty xe^{-x} \coth \frac{x}{2} \, dx = \frac{\pi^2}{3} - 1$$

Write the integral as

$$\int_0^\infty xe^{-x} \coth \frac{x}{2} \, dx = \int_0^\infty \frac{xe^{-x} \, dx}{1 - e^{-x}} + \int_0^\infty \frac{xe^{-2x} \, dx}{1 - e^{-x}}.$$

The integral representation

$$\zeta(z, q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1} e^{-qt}}{1 - e^{-t}} \, dt$$

gives

$$\int_0^\infty xe^{-x} \coth \frac{x}{2} \, dx = \Gamma(2)\zeta(2, 1) + \Gamma(2)\zeta(2, 2).$$

The series expression for the Hurwitz zeta function

$$\zeta(z, q) = \sum_{n=0}^\infty \frac{1}{(n+q)^z}$$

gives the result.