PROOF OF FORMULA 3.552.6

$$\int_0^\infty \frac{x^3 e^{-2nx}}{\sinh x}\,dx = \frac{\pi^4}{8} - 12\sum_{k=1}^{n} \frac{1}{(2k-1)^4}$$

Write the integral as

$$\int_0^\infty \frac{x^3 e^{-2nx}}{\sinh x}\,dx = 2\int_0^\infty \frac{x^3 e^{-(2n+1)x}\,dx}{1-e^{-2x}}.$$

The change of variables $t = 2x$ gives

$$\int_0^\infty \frac{x^3 e^{-2nx}}{\sinh x}\,dx = \frac{1}{8}\int_0^\infty \frac{t^3 e^{-(2n+1)t}}{1-e^{-t}}.$$

Expand the denomintaor of the integrand as a geometric series to obtain

$$\int_0^\infty \frac{x^3 e^{-2nx}}{\sinh x}\,dx = 12\sum_{k=n}^{\infty} \frac{1}{(2k+1)^4}.$$

This is the result using

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^2}{96}.$$