

**PROOF OF FORMULA 3.553.1**

$$\int_0^\infty \frac{\sinh^2(ax)}{\sinh x} \frac{e^{-x}}{x} dx = \frac{1}{2} \ln \left( \frac{\pi a}{\sin \pi a} \right)$$

Use the change of variables  $t = 2x$  to write the integral as

$$\int_0^\infty \frac{\sinh^2(ax)}{\sinh x} \frac{e^{-x}}{x} dx = \frac{1}{2} \int_0^\infty \frac{e^{(a-1)t}(1 - e^{-at})^2}{1 - e^{-t}} \frac{dt}{t}.$$

Entry 3.413.1 states that

$$\int_0^\infty \frac{(1 - e^{-ax})(1 - e^{-bx})e^{-cx}}{1 - e^{-x}} \frac{dx}{x} = \ln \left[ \frac{\Gamma(c)\Gamma(a+b+c)}{\Gamma(a+c)\Gamma(b+c)} \right].$$

Employ this formula with  $c = 1 - a$  and  $b = a$  and use the relation

$$\Gamma(a)\Gamma(1-a) = \frac{\pi a}{\sin \pi a}$$

to produce the stated answer.