

PROOF OF FORMULA 3.562.3

$$\int_0^{\infty} x e^{-\beta x^2} \sinh \gamma x \, dx = \frac{\gamma}{4\beta} \sqrt{\frac{\pi}{\beta}} e^{\gamma^2/4\beta}$$

Write the integral as

$$\begin{aligned} \int_0^{\infty} x e^{-\beta x^2} \sinh \gamma x \, dx &= \frac{1}{2} \int_0^{\infty} x e^{-\beta x^2 + \gamma x} \, dx - \frac{1}{2} \int_0^{\infty} x e^{-\beta x^2 - \gamma x} \, dx \\ &= \frac{1}{2} e^{\gamma^2/4\beta} \int_{-\gamma/2\beta}^{\infty} \left(t + \frac{\gamma}{2\beta}\right) e^{-\beta t^2} \, dt - \frac{1}{2} e^{\gamma^2/4\beta} \int_{\gamma/2\beta}^{\infty} \left(t - \frac{\gamma}{2\beta}\right) e^{-\beta t^2} \, dt \\ &= \frac{1}{2} e^{\gamma^2/4\beta} \int_{-\gamma/2\beta}^{\infty} t e^{-\beta t^2} \, dt + \frac{\gamma}{4\beta} e^{\gamma^2/4\beta} \int_{-\gamma/2\beta}^{\infty} e^{-\beta t^2} \, dt \\ &\quad - \frac{1}{2} e^{\gamma^2/4\beta} \int_{\gamma/2\beta}^{\infty} t e^{-\beta t^2} \, dt + \frac{\gamma}{4\beta} e^{\gamma^2/4\beta} \int_{\gamma/2\beta}^{\infty} e^{-\beta t^2} \, dt. \end{aligned}$$

The first and third integral cancel each other. The second and fourth give

$$\begin{aligned} \int_0^{\infty} x e^{-\beta x^2} \sinh \gamma x \, dx &= \frac{\gamma}{4\beta} e^{\gamma^2/4\beta} \left(\int_{-\gamma/2\beta}^{\infty} e^{-\beta t^2} \, dt + \int_{\gamma/2\beta}^{\infty} e^{-\beta t^2} \, dt \right) \\ &= \frac{\gamma}{4\beta} e^{\gamma^2/4\beta} \int_{-\infty}^{\infty} e^{-\beta t^2} \, dt \\ &= \frac{\gamma}{4\beta} \sqrt{\frac{\pi}{\beta}} e^{\gamma^2/4\beta}, \end{aligned}$$

as claimed.