

PROOF OF FORMULA 3.661.2

$$\int_0^{2\pi} (a \sin x + b \cos x)^{2n} dx = \frac{(2n-1)!!}{(2n)!!} 2\pi (a^2 + b^2)^n$$

Let $u = 2\pi - x$ to obtain

$$\int_0^{2\pi} (a \sin x + b \cos x)^{2n} dx = \int_0^{2\pi} (-a \sin x + b \cos x)^{2n} dx.$$

Expand using the binomial theorem and add the two expressions to obtain

$$2 \int_0^{2\pi} (a \sin x + b \cos x)^{2n} dx = 2 \int_0^{2\pi} \sum_{j=0}^n \binom{2n}{2j} a^{2j} b^{2n-2j} \sin^{2j} x \cos^{2n-2j} x dx.$$

Conclude that

$$\int_0^{2\pi} (a \sin x + b \cos x)^{2n} dx = 4 \sum_{j=0}^n \binom{2n}{2j} a^{2j} b^{2n-2j} \int_0^{\pi/2} \sin^{2j} x \cos^{2n-2j} x dx.$$

The integral is

$$\int_0^{\pi/2} \sin^{2j} x \cos^{2n-2j} x dx = \frac{1}{2} B\left(j + \frac{1}{2}, n - j + \frac{1}{2}\right).$$

The result follows from

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} (2n-1)!!,$$

and it can also be written as

$$\int_0^{2\pi} (a \sin x + b \cos x)^{2n} dx = \frac{2\pi}{2^{2n}} \binom{2n}{n} (a^2 + b^2)^n.$$