

**PROOF OF FORMULA 4.223.2**

$$\int_0^{\infty} \ln(1 - e^{-x}) dx = -\frac{\pi^2}{6}$$

Start with the expansion

$$\ln(1 - t) = -\sum_{k=0}^{\infty} \frac{t^{k+1}}{k+1}$$

to obtain

$$\ln(1 - e^{-x}) = -\sum_{k=0}^{\infty} \frac{e^{-(k+1)x}}{k+1}.$$

Integration produces

$$\int_0^{\infty} \ln(1 - e^{-x}) dx = -\sum_{k=0}^{\infty} \frac{1}{k+1} \int_0^{\infty} e^{-(k+1)x} dx.$$

The change of variables  $t = (k+1)x$  gives the result

$$\int_0^{\infty} \ln(1 - e^{-x}) dx = -\sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = -\frac{\pi^2}{6}.$$