PROOF OF FORMULA 4.224.2

$$\int_0^{\pi/4} \ln \sin x \, dx = -\frac{\pi}{4} \ln 2 - \frac{G}{2}$$

The Catalan constant has the integral representation

$$G = -\int_0^{\pi/4} \ln \tan x \, dx$$

given in 4.227.2. This gives

$$\int_0^{\pi/4} \ln \sin x \, dx - \int_0^{\pi/4} \ln \cos x \, dx = -G.$$

On the other hand,

$$\begin{aligned} \int_0^{\pi/4} \ln \sin x \, dx + \int_0^{\pi/4} \ln \cos x \, dx &= \int_0^{\pi/4} \ln(\sin x \cos x) \, dx \\ &= \int_0^{\pi/4} \ln \left(\frac{\sin 2x}{2} \right) \, dx \\ &= \frac{1}{2} \int_0^{\pi/2} \ln \sin t \, dt - \frac{\pi}{4} \ln 2. \end{aligned}$$

Using the evaluation

$$\int_0^{\pi/2} \ln \sin t \, dt = -\frac{\pi}{2} \ln 2$$

given in 4.224.3 produces

$$\int_0^{\pi/4} \ln \sin x \, dx + \int_0^{\pi/4} \ln \cos x \, dx = -\frac{\pi}{2} \ln 2.$$

Solving the system of equation for the two unknown integrals gives

$$\int_0^{\pi/4} \ln \sin x \, dx = -\frac{\pi}{4} \ln 2 - \frac{G}{2},$$

and

$$\int_0^{\pi/4} \ln \cos \, dx = -\frac{\pi}{4} \ln 2 + \frac{G}{2},$$