PROOF OF FORMULA 4.224.9

$$\int_0^\pi \ln(a + b\cos x)\, dx = \pi \ln\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right)$$

Let $I(a, b)$ be the integral above. Then the transformation $u = \tan(x/2)$ yields $\cos x = (1 - u^2)/(1 + u^2)$ and $dx = 2du/(1 + u^2)$. Thus

$$\frac{\partial I}{\partial b} = \int_0^\pi \frac{\cos x\, dx}{a + b\cos x} = \frac{2}{a - b} \int_0^\infty \frac{1 - u^2}{(1 + u^2)(u^2 + c)}$$

with $c = (a + b)/(a - b)$. The method of partial fractions gives

$$\frac{\partial I}{\partial b} = \frac{\pi}{b} - \frac{\pi a}{b\sqrt{a^2 - b^2}}.$$

Integrating with respect to b yields

$$I(a, b) = \pi \ln 2 + \pi \ln(a + \sqrt{a^2 - b^2}) + C(a).$$

The value $I(a, 0) = \pi \ln a$ evaluates the constant of integration $C(a)$ and provides the final result.