PROOF OF FORMULA 4.227.15

$$\int_0^{\pi/4} \ln(\tan x + \cot x)\,dx = \frac{1}{2}\int_0^{\pi/2} \ln(\tan x + \cot x)\,dx = \frac{\pi}{2}\ln 2$$

The change of variables $t = \pi/2 - x$ proves the first equality. To obtain the value of the integral, write it as

$$\int_0^{\pi/2} \ln(\tan x + \cot x)\,dx = -\int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right)\,dx.$$

The change of variables $t = 2x$ gives

$$\int_0^{\pi/2} \ln(\tan x + \cot x)\,dx = \frac{\pi}{2}\ln 2 - \frac{1}{2}\int_0^{\pi} \ln\sin t\,dt.$$

Symmetry now gives

$$\int_0^{\pi/2} \ln(\tan x + \cot x)\,dx = \frac{\pi}{2}\ln 2 - \int_0^{\pi/2} \ln\sin t\,dt$$

and the result follows from entry 4.224.3

$$\int_0^{\pi/2} \ln\sin t\,dt = -\frac{\pi}{2}\ln 2.$$